GOCE gravity field model derived from rotational invariants

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Session 2: Schwerfeld und Geoid
Geodätische Woche 2011
27.–29. September 2011, Messe Nürnberg
Main Topics

1. GOCE invariant observable
2. Invariant analysis method
3. Results and analysis
4. Conclusions and further studies
1. GOCE invariant observable

**GOCE gradiometry**

$\Gamma = -V + \Omega^2 + \dot{\Omega}$ → separation of centrifugal and Euler effects (star tracker, gradiometer)

$\Gamma$ ... observation tensor

$V$ ... gravitational tensor

$\Omega^2$ ... centrifugal tensor

$\dot{\Omega}$ ... Euler tensor

Gravitational gradients:

$$V = V_{ij} = \begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{xy} & V_{yy} & V_{yz} \\ V_{xz} & V_{yz} & V_{zz} \end{bmatrix}, \quad V_{xx} + V_{yy} + V_{zz} = 0$$
high-sensitive axes accuracy: $10^{-12}$ m s$^{-2}$
low-sensitive axes accuracy: $10^{-9}$ m s$^{-2}$

→ high-accurate determination of $V_{xx}, V_{yy}, V_{zz}, V_{xz}$
GOCE EGG Invariant representation

Standard approach:
observations: $V_{ij}$
gradiometer orientation essential (tensor transformation)

Alternative approach:
observations: $J = J\{V\} = J\{V_{ij}\}$
no orientation information required
→ rotational invariants

gradiometer frame (GRF)
\[
\{ e_1^G , e_2^G , e_3^G \mid 0^* \}
\]
⇔
model frame (LNOF)
\[
\{ e_1^M , e_2^M , e_3^M \mid 0 \}
\]
Complete system consists of three independent invariants

System I
Traces of matrix product

\[ J_1 = \text{tr } V \]
\[ J_2 = \text{tr } V^2 \]
\[ J_3 = \text{tr } V^3 \]

System II
Sums of principal minor determinants

\[ I_1 = \text{tr } V \]
\[ I_2 = \frac{1}{2} [(\text{tr } V)^2 - \text{tr } V^2] \]
\[ I_3 = \det V \]

System III
Eigenvalues

\[ \Lambda_1 \]
\[ \Lambda_2 \]
\[ \Lambda_3 \]

Waring formula

Newton-Girard formula

characteristic equation

\[ p(\Lambda) = \det( V - \Lambda I_3) = 0 \]
\[ \Lambda^3 - I_1 \Lambda^2 + I_2 \Lambda - I_3 = 0 \]
For a symmetric second-order tensor these invariants simplify to

\[ I_1 = V_{xx} + V_{yy} + V_{zz}, \quad I_1 = \Lambda_1 + \Lambda_2 + \Lambda_3, \]
\[ I_2 = V_{xx}V_{yy} + V_{xx}V_{zz} + V_{yy}V_{zz} - V_{xy}^2 - V_{xz}^2 - V_{yz}^2, \quad I_2 = \Lambda_1 \Lambda_2 + \Lambda_1 \Lambda_3 + \Lambda_2 \Lambda_3, \]
\[ I_3 = V_{xx}V_{yy}V_{zz} + 2V_{xy}V_{xz}V_{yz} - V_{xx}V_{yy}^2 - V_{xx}V_{yz}^2 - V_{xz}V_{xz}V_{yz}^2, \quad I_3 = \Lambda_1 \Lambda_2 \Lambda_3. \]

- analysis of \( I_1 \) provides the trivial solution (\( \rightarrow \) constraints, but not trace-free!)
- non-linear gravity field functionals
- gravitational gradients products
- mixing of gravitational gradients
Synthesis of unobserved GGs

- Invariants representation requires the GGs with compatible accuracy (full tensor gradiometry)
- GOCE: $V_{xz}$ and $V_{yz}$ highly reduced in accuracy
- Synthetic calculation of inaccurate GGs (forward modeling)
- Avoid a priori information to leak into gravity field estimate
- $V_{xy}, V_{yz} \rightarrow V_{xx}, V_{yy}, V_{zz} \rightarrow$ minor influence

Additional effort: synthesis of $V_{xz}$ and $V_{yz}$
2. Invariant analysis method

- GOCE EGG Invariant parameterization

- The spherical harmonic gravitational potential representation:
  \[ V(r, \theta, \lambda) = \frac{GM}{R} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{m=0}^{n} P_{nm} \left( \cos \theta \right) \left( C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right) \]

- The second order gravitational tensor:
  \[ \mathbf{V} = \nabla \nabla V(r, \theta, \lambda) = e_i \otimes e_j V_{ij} \]

- For example, the gravity gradient of zz component:
  \[ V_{zz}(r, \theta, \lambda) = \frac{GM}{R^3} \sum_{n=0}^{N_{\text{max}}} (n+1)(n+2) \left( \frac{R}{r} \right)^{n+3} \sum_{m=0}^{n} P_{nm} \left( \cos \theta \right) \left( C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right) \]

- The second rotational invariant:
  \[ I_2 = V_{xx} V_{yy} + V_{xx} V_{zz} + V_{yy} V_{zz} - V_{xy}^2 - V_{xz}^2 - V_{yz}^2 \]
GOCE EGG Invariant linearization

Linearization through disturbing linearization theory (DTL), or the so called, calculation of perturbations: \( V_{ij} = U_{ij} + T_{ij} \)

\[
\delta I_2 = I_2 - I_2^{\text{ref}} = U_{xx} T_{yy} + U_{xx} T_{zz} + U_{yy} T_{zz} + U_{yy} T_{xx} + U_{zz} T_{xx} + U_{zz} T_{yy} \\
-2(U_{xy} T_{xy} - U_{xz} T_{xz} - U_{yz} T_{yz}) + O^2(T_{ij})
\]

\[
\delta I_3 = I_3 - I_3^{\text{ref}} = U_{xx} U_{yy} T_{zz} + U_{xx} T_{yy} U_{zz} + T_{xx} U_{yy} U_{zz} + 2(U_{xy} U_{xz} T_{yz} + U_{xy} T_{xz} U_{yz} + T_{xy} U_{xz} U_{yz}) \\
-T_{xx} U_{yz}^2 - T_{yy} U_{xz}^2 - T_{zz} U_{xy}^2 - 2(U_{xx} U_{yz} T_{yz} - U_{yy} U_{xz} T_{xz} - U_{zz} U_{xy} T_{xy}) \\
+O^2(T_{ij}) + O^3(T_{ij})
\]

additional effort (per iteration): synthesis of \( U_{ij} \) up to \( L^{\text{ref}} \leq L \)
The statistical model

- The linearized invariant $I_2$:

$$I_2 = c_1 T_{xx} + c_2 T_{xy} + c_3 T_{xz} + c_4 T_{yy} + c_5 T_{yz} + c_6 T_{zz} + I_{2\text{ref}}$$

- Variance-covariance matrix of the invariant $I_2$:

$$D(I_2) = JD(V)J^T$$

$$D(V) – \text{the total GGs variance-covariance matrix;}$$

- Neglecting correlations among GGs, $D(I_2)$:

$$D(I_2) = J_1 D(V_{xx})J_1^T + \cdots + J_6 D(V_{zz})J_6^T =$$

$$= J_1 D(T_{xx})J_1^T + \cdots + J_6 D(T_{zz})J_6^T$$

Inserting $D(V_{ij}) = (F_{V_{ij}}^T F_{V_{ij}})^{-1}$, where $F_{V_{ij}}$ ARMA Filters, yielding

$$D(I_2) = J_1 F_{V_{xx}}^{-1}(J_1 F_{V_{xx}}^{-1})^T + \cdots + J_6 F_{V_{zz}}^{-1}(J_6 F_{V_{zz}}^{-1})^T.$$  

But the inverse of $D(I_2)$?
The statistical model

- For one month GOCE observations with rate 1 Hz, the matrix size of $D(V)$: 1.76 PetaByte

- There are 2592000 invariants $I_2$ and the matrix size of $D(I_2)$ is 48.88 TeraByte, and the inverse of $D(I_2)$?

- Not possible to simply deduce the invariants variance-covariance information from the gravitational gradients by error propagation.
- The time series of these invariants have similar stochastic characters to these gravitational gradients.
- Based on the successful experiences from Bonn group (Schuh et al.), the GOCE invariants can be considered as equidistant time series and therefore be decorrelated by means of numerical filters.
Statistical Study of rotational invariants

Time series of REAL GOCE gravity gradients EGG NOM and three invariants (in red) from 02-Nov-2009 to 31-Dec-2009
Global map of the Invariant $I_2$ from 02-Nov-2009 to 31-Dec-2009, in $E^2$
- measured mean & GRS reduced Vzz gradient on 02. Nov. 2009 (4 hours)
- computed mean & GRS reduced Vzz gradient (EIGEN05C)
- estimated mean reduced Vzz noise (measured-computed)
- signal and noise have the same order of magnitude

- measured mean & GRS reduced Invariant $I_2$ on 02. Nov. 2009 (4 hours)
- computed mean & GRS reduced Invariant $I_2$ (EIGEN05C)
- estimated mean reduced Invariant $I_2$ noise (measured-computed)
- the time series of Invariant $I_2$ has similar stochastic characters to GOCE gradients
- measured mean & GRS reduced Invariant $I_3$ on 02. Nov. 2009 (4 hours)
- computed mean & GRS reduced Invariant $I_3$ (EIGEN05C)
- estimated mean reduced Invariant $I_3$ noise (measured-computed)
- the time series of Invariant $I_3$ has similar stochastic characters to GOCE gradients
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- GOCE Invariant $I_2$ residuals
  (measured – computed)
  from 02-Nov-2009 to 31-Dec-2009

- GOCE Invariant $I_3$ residuals
  (measured – computed)
  from 02-Nov-2009 to 31-Dec-2009
Statistical study of rotational invariants

PSD of GOCE Invariant $I_2$ for Nov. and Dec. 2009

PSD of GOCE Invariant $I_3$ for Nov. and Dec. 2009
Statistical study of rotational invariants

PSD of GOCE Invariant $I_1$ for Nov. and Dec. 2009
Filter equation – MA(q)

\[ y_t = \sum_{k=0}^{q} \beta_k u_{t-k}, \]

Filter cascade:
PSD of GOCE Invariant $I_2$ residuals, High pass filtered and MA filtered residuals for Nov. and Dec. 2009. (with the support from ITG Bonn)
Application of decorrelation filters to GOCE invariant observation model:

\[ \mathbf{F}_i \mathbf{y}_i = \mathbf{F}_i \mathbf{A}_i \mathbf{x} + \mathbf{F}_i \mathbf{e}_i \]

And filtered GOCE invariant observation model:

\[ \overline{\mathbf{y}}_i = \overline{\mathbf{A}}_i \mathbf{x} + \overline{\mathbf{e}}_i, \quad \text{with} \quad \Sigma_{\overline{\mathbf{y}}_i} = \mathbf{I}. \]

Processing:

1. the polar gap problem is solved through the order-dependent Kaula regularization with a proper regularization parameter.

\[
\mathbf{x} = (\overline{\mathbf{A}}_i^T \overline{\mathbf{A}}_i + \alpha \mathbf{R})^{-1} \overline{\mathbf{A}}_i^T \overline{\mathbf{y}}_i
\]

where \( r_{ij} = \begin{cases} n^4 & \text{if } i = j \text{ and } m \leq m_{\text{reg}} (= 15) \\ 0 & \text{otherwise} \end{cases} \)

2. High performance computing with OpenMP and MPI.
Signal degree amplitudes and RMS degree errors of GOCE Invariant $I_2$ solution (EIGEN05C as reference model in linearization of invariants) with respect to the EGM08 model, which show that this GOCE invariant solution is consistent to the three GOCE Solutions published in July 2010.
Coefficient differences of GOCE invariant I2 solution compared to EIGEN50C
Geoid Differences (in meter) between GOCE Invariant I2 solution and EGM08 model (substituted with GOCE TIM1 lower o/d until 50)

<table>
<thead>
<tr>
<th>sector</th>
<th>min</th>
<th>max</th>
<th>mean</th>
<th>rms</th>
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<td>3.671</td>
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<td>90.0</td>
<td>-3.795</td>
<td>5.402</td>
<td>0.061</td>
<td>0.688</td>
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Further GOCE invariant analysis result:

Signal degree amplitudes and RMS degree errors of GOCE Invariant $I_2$ solution (GIS GOCE SST solution (d/o 110) as reference model in linearization of invariants) with respect to the EGM08 model, which show that this GOCE invariant analysis result is also consistent to the three GOCE Solutions published in July 2010.
4. Conclusions and further studies

- The cascade filters have been estimated and implemented in GOCE invariant analysis;

- The polar gap problem is solved through the order-dependent Kaula regularization with a proper regularization parameter;

- One high-resolution global gravity field model until degree/order 224 has been derived based on two months of GOCE invariant data;

- The RMS degree variances show that GOCE invariant solution is consistent to the three GOCE Solutions published in July 2010, which complementarily supports the GOCE combination solutions;

- With our GOCE SST solution (d/o 110) as reference model in linearization, one GOCE-only global gravity field model is also estimated with GOCE invariants.

- Further GOCE invariant solutions will be performed with longer period GOCE observations.
Thank you!