Characteristics of different preconditioners used in GOCE gravity field determination applying iterative solvers

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1 Motivation

2 Design of preconditioners for GOCE

3 Comparing different preconditioners
   - For CG acceleration
   - For estimation of spherical harmonics standard deviations
   - For variance propagation

4 Summary and Outlook
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Tuning-Machine within ESA-HPF – Time-Wise solution

- iterative refinement of stochastic model of SGG observations
- estimation of relative weights $\omega_i$
- iterative outlier flagging
- non-approximative solution for gravity field coefficients
- approximative accuracy information ($\hat{\Sigma}_{xx}$, $\sigma_{c,s lm}$)

$\Rightarrow$ repeated assembling and solution of

$$
\begin{pmatrix}
\omega_{sst} N_{sst} + \omega_{sgg} A_{sgg}^T F_{sgg}^T F_{sgg} A_{sgg} + \omega_{reg} N_{reg}
\end{pmatrix} x = \\
\begin{pmatrix}
\omega_{sst} n_{sst} + \omega_{sgg} A_{sgg}^T F_{sgg}^T l_{sgg} + \omega_{reg} n_{reg}
\end{pmatrix}
$$

$N x = n$

$\Rightarrow$ needs to be assembled $\approx 10$ times and solved $\approx 30$ times

$\Rightarrow$ fast, iterative, tailored Conjugate Gradients based solver to avoid computation of $A_{sgg}^T F_{sgg}^T F_{sgg} A_{sgg}$
Gravity field determination from GOCE

Tuning-Machine within ESA-HPF – Time-Wise solution

- iterative refinement of stochastic model of SGG observations
- estimation of relative weights $\omega_i$
- iterative outlier flagging
- non-approximative solution for gravity field coefficients
- approximative accuracy information $(\hat{\Sigma}_{xx}, \sigma_{\oplus,c,s_{lm}})$

⇒ repeated assembling and solution of

$$\left(\omega_{sst}N_{sst} + \omega_{sgg}A_{sgg}^T F_{sgg}^T F_{sgg} A_{sgg} + \omega_{reg}N_{reg}\right) x = \omega_{sst}n_{sst} + \omega_{sgg}A_{sgg}^T F_{sgg}^T l_{sgg} + \omega_{reg}n_{reg}$$

$N x = n$

⇒ needs to be assembled $\approx 10$ times and solved $\approx 30$ times

⇒ fast, iterative, tailored Conjugate Gradients based solver to avoid computation of $A_{sgg}^T F_{sgg}^T F_{sgg} A_{sgg}$
Although PCGMA works on SGG observation equations and REG, SST normal equations, the convergence of Conjugate Gradients depends on the spectral properties of $N$

Eigenvalues of the combined normal equation matrix $N$ of the Release 2 Time-Wise GOCE model (do 2-250)
Convergence behaviour of CG

Although PCGMA works on SGG observation equations and REG, SST normal equations, the convergence of Conjugate Gradients depends on the spectral properties of \( \mathbf{N} \) eigenvalues:

\[
\lambda_{\text{min}} = 1.7854 \cdot 10^{17} \\
\lambda_{\text{max}} = 6.0909 \cdot 10^{23}
\]

condition:

\[
\kappa = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} = 3.4166 \cdot 10^6
\]

spectral radius:

\[
\rho_{\text{CG}} = \left( \frac{1 - \sqrt{\kappa}}{1 + \sqrt{\kappa}} \right)^2 = 0.9978
\]

\# iterations for \( d = 6 \) significant digits:

\[
\nu = \left| \frac{d}{\log_{10}\rho_{\text{CG}}} \right| > 6000
\]

Eigenvalues of the combined normal equation matrix \( \mathbf{N} \) of the Release 2 Time-Wise GOCE model (do 2-250)
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Accelerating CG convergence

Preconditioning to:
- accelerate CG convergence
- obtain approx. accuracy information (covariance matrix)

Design of preconditioner (representative matrix) for GOCE:
- sparse matrix $N \oplus$, reflecting main characteristics of $N$
- no-fill in values during Cholesky factorization $R \oplus = \text{chol}(N \oplus)$
  $R \oplus$ should have same sparsity pattern as triang. part of $N \oplus$
Preconditioning to:

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Design of preconditioner (representative matrix) for GOCE:

- sparse matrix $\mathbf{N}_\oplus$, reflecting main characteristics of $\mathbf{N}$
- no-fill in values during Cholesky factorization $\mathbf{R}_\oplus = \text{chol}(\mathbf{N}_\oplus)$
  - $\mathbf{R}_\oplus$ should have same sparsity pattern as triang. part of $\mathbf{N}_\oplus$
Accelerating CG convergence

Preconditioning to:

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- obtain approx. accuracy information (covariance matrix)

Design of preconditioner (representative matrix) for GOCE:

- sparse matrix \( N_\oplus \), reflecting main characteristics of \( N \)
- no-fill in values during Cholesky factorization \( R_\oplus = \text{chol} (N_\oplus) \)

\( R_\oplus \) should have same sparsity pattern as triang. part of \( N_\oplus \)

⇒ use orthogonalities of Legendre functions
⇒ known dominant structures of normals
⇒ equatorial symmetry
⇒ special numbering schemes
Preconditioners used for GOCE

For GOCE different preconditioners are applied:

\[ \mathbf{N}_\oplus = \omega_{\text{SST}} \cdot \text{selectEntries}(\mathbf{N}_{\text{SST}}) + \omega_{\text{SGG}} \cdot \mathbf{N}_{\oplus,\text{SGG}} + \omega_{\text{REG}} \cdot \mathbf{N}_{\text{REG}} \]

- GPS tracking (SST)
  - Fully available, relatively small
  - \(\Rightarrow\) a) added without approximation to \(\mathbf{N}_\oplus\)
  - \(\Rightarrow\) b) Reduced to block diagonal pattern

- Gradiometry (SGG)
  - Unknown, computation avoided
  - \(\Rightarrow\) define sparsity mask for \(\mathbf{N}_{\oplus,\text{SGG}}\)
  - \(\Rightarrow\) compute non-zero elements \(\mathbf{N}_{\oplus,\text{SGG}}(i,j) = \mathbf{A}_{\text{SGG}}(:,i)^T \mathbf{F}_{\text{SGG}}^T \mathbf{F}_{\text{SGG}} \mathbf{A}_{\text{SGG}}(:,j)\)

- Prior information (REG)
  - Diagonal, fully available
  - \(\Rightarrow\) added without approximation to \(\mathbf{N}_\oplus\)

Using this strategy it is not guaranteed that \(\mathbf{N}_\oplus\) is pos. definite!
Preconditioners used for GOCE

For GOCE different preconditioners are applied:

\[ \mathbf{N} \oplus = \omega_{\text{SST}} \cdot \text{selectEntries}(\mathbf{N}_{\text{SST}}) + \omega_{\text{SGG}} \cdot \mathbf{N}_{\oplus,\text{SGG}} + \omega_{\text{REG}} \cdot \mathbf{N}_{\text{REG}} \]

**GPS tracking (SST)**
- Fully available, relatively small
- \(\Rightarrow\) a) added without approximation to \(\mathbf{N} \oplus\)
- \(\Rightarrow\) b) Reduced to block diagonal pattern

**Gradiometry (SGG)**
- Unknown, computation avoided
- \(\Rightarrow\) define sparsity mask for \(\mathbf{N}_{\oplus,\text{SGG}}\)
- \(\Rightarrow\) compute non-zero elements
  \[ \mathbf{N}_{\oplus,\text{SGG}}(i,j) = \mathbf{A}_{\text{SGG}}(:,i)^T \mathbf{F}_{\text{SGG}}^T \mathbf{F}_{\text{SGG}} \mathbf{A}_{\text{SGG}}(:,j) \]

**Prior information (REG)**
- Diagonal, fully available
- \(\Rightarrow\) added without approximation to \(\mathbf{N} \oplus\)

Using this strategy it is not guaranteed that \(\mathbf{N} \oplus\) is pos. definite!
Symbolic shape preconditioners

01: bdSST + bdSGG + dREG
4 blocks/order

02: bdSST + bdSGG + dREG
2 blocks/order

03: fSST + bdSGG + dREG
kite numbering

04: fSST + fsSGG + dREG
4 blocks/order

05: fSST + fsSGG + dREG
free kite numbering

06: fSST + fsSGG + dREG
free kite numbering

bd: blockdiagonal, d: diagonal, f: full, fs: all values of sparse pattern
Evaluated preconditioners

For GOCE RL02 data using the time-wise method:
Full normals $\mathbb{N}$ (d/o 2-250) as reference.

<table>
<thead>
<tr>
<th>ID</th>
<th>SST</th>
<th>SGG</th>
<th>ns</th>
<th>entries</th>
<th>mem [GB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1_bd+bd_sym</td>
<td>bd</td>
<td>bd</td>
<td>order</td>
<td>5 270 253</td>
<td>0.039</td>
</tr>
<tr>
<td>2_bd+bd</td>
<td>bd</td>
<td>bd</td>
<td>order</td>
<td>10 540 253</td>
<td>0.079</td>
</tr>
<tr>
<td>3_full+bd</td>
<td>full</td>
<td>bd</td>
<td>kite</td>
<td>113 832 959</td>
<td>0.848</td>
</tr>
<tr>
<td>4_kite</td>
<td>full</td>
<td>bd+l ≤ 100</td>
<td>fkn</td>
<td>113 832 959</td>
<td>0.848</td>
</tr>
<tr>
<td>5_freekite10</td>
<td>full</td>
<td>bd+l ≤ 100+m ≤ 10</td>
<td>fkn</td>
<td>186 921 959</td>
<td>1.393</td>
</tr>
<tr>
<td>6_freekite30</td>
<td>full</td>
<td>bd+l ≤ 100+m ≤ 30</td>
<td>fkn</td>
<td>381 219 959</td>
<td>2.840</td>
</tr>
</tbody>
</table>

⇒ using the corresponding numberings, none of the preconditioners produce fill-in elements during Cholesky!

bd: blockdiagonal, fkn: free kite numbering, sym: assuming equator symmetry (odd/even orders uncorrelated)
Motivation

Design of preconditioners for GOCE

Comparing different preconditioners

- For CG acceleration
- For estimation of spherical harmonics standard deviations
- For variance propagation

Summary and Outlook
Proconditioning

Instead of

\[ Nx = n \]

the preconditioned system

\[ N_{\oplus}^{-1}Nx = N_{\oplus}^{-1}n \]
\[ \tilde{N}x = \tilde{n} \]

is solved. \( \tilde{N} \) is in general not symmetric or pos. definite! But

\[ R_{\oplus}^{-T}NR_{\oplus}^{-1}R_{\oplus}x = R_{\oplus}^{-T}n \]
\[ \tilde{N}\tilde{x} = \tilde{n} \]

has same spectral properties and is solved with CG. \( R_{\oplus}^{-T}NR_{\oplus}^{-1} \) is symmetric and pos. def.!

Notes:

▶ in CG/PCGMA the preconditioning in be performed by the preconditioning of residuals: \( N_{\oplus}^{-1}r = N_{\oplus}^{-1}Nx - n \)

▶ \( N_{\oplus}^{-1}N \) has same eigenvalues then \( R_{\oplus}^{-T}NR_{\oplus}^{-1} \)!

▶ for details see Schuh (1996, p. 82ff)
Using HPC, the preconditioned matrix can be explicitly computed for purpose of analysing the efficiency of the preconditioners: e.g. eigenvalues and condition of $\tilde{N}$

Estimated eigenvalues & condition of system which is solved for with PCGMA ($\tilde{N}$), $\kappa(N) = 3.4 \cdot 10^6$

<table>
<thead>
<tr>
<th>ID</th>
<th>$\lambda_{\text{min}}$</th>
<th>$\lambda_{\text{max}}$</th>
<th>$\kappa$</th>
<th>$\rho_{CG}$</th>
<th>$\nu(d = 6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1_bbd+bd_sym</td>
<td>0.151</td>
<td>1.911</td>
<td>12.67785</td>
<td>0.31524</td>
<td>12</td>
</tr>
<tr>
<td>2_bbd+bd</td>
<td>0.483</td>
<td>1.290</td>
<td>2.66969</td>
<td>0.05792</td>
<td>4</td>
</tr>
<tr>
<td>3_full+bd</td>
<td>0.483</td>
<td>1.290</td>
<td>2.66963</td>
<td>0.05792</td>
<td>4</td>
</tr>
<tr>
<td>4_kite</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>not pos. def.!</td>
</tr>
<tr>
<td>5_freekite10</td>
<td>0.484</td>
<td>1.283</td>
<td>2.65143</td>
<td>0.05715</td>
<td>4</td>
</tr>
<tr>
<td>6_freekite30</td>
<td>0.484</td>
<td>1.282</td>
<td>2.65141</td>
<td>0.05715</td>
<td>4</td>
</tr>
</tbody>
</table>
Standard deviations of SH coefficients

To obtain accuracy information using the iterative solver, use inverse of the preconditioner

\[ \Sigma_{\oplus} = N_{\oplus}^{-1}. \]

But, fill-in elements occur, \( \Sigma_{\oplus} \) is not sparse, it is full! Use partial inverse \( \hat{\Sigma}_{\oplus} \), which has same sparsity pattern as \( N_{\oplus} \)

\[ \hat{\Sigma}_{\oplus}(i, j) = \Sigma_{\oplus}(i, j), \text{ if } N_{\oplus} \neq 0, \]
\[ \hat{\Sigma}_{\oplus}(i, j) \neq \Sigma_{\oplus}(i, j), \text{ if } N_{\oplus} = 0, \]

to determine the diagonal of \( N_{\oplus}^{-1} \) as standard deviations of sh:

\[ \text{diag} (\hat{\Sigma}_{\oplus}) = \text{diag} (\Sigma_{\oplus}) = \text{diag} (N_{\oplus}^{-1}) \]

Compare the standard deviations \( \sigma_{c,s lm} \) obtained from inversion of full normals \( \Sigma = N^{-1} \) to standard deviations estimated from the preconditioners \( \sigma_{\oplus,c,s lm} \) to evaluate different preconditioners.
From the inversion of the full normal equations “true” standard deviations are known: $\Sigma = N^{-1} \Rightarrow \sigma_{c,s_{lm}}$

Standard deviations of the GOCE_EGM_TIM_RL0002 gravity field model. Shown is $\log_{10}(\sigma_{c,s_{lm}})$ in the coefficient triangle.
Relative errors of standard deviations

\[
\Delta_{lm} = \frac{\sigma_{\oplus,c,s_{lm}} - \sigma_{c,s_{lm}}}{\sigma_{c,s_{lm}}}
\]

Relative error of standard deviation

\[
\log_{10} |\Delta_{lm}|
\]

<table>
<thead>
<tr>
<th>ID</th>
<th>(\Delta_{lm}) %</th>
<th>max (\Delta_{lm}) %</th>
<th>mean (\Delta_{lm}) %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-30.1</td>
<td>-0.0</td>
<td>-2.54</td>
</tr>
</tbody>
</table>

Nearly all differences negative, \(\sigma_{\oplus,c,s_{lm}}\) is always too optimistic!
Relative errors of standard deviations

\[ \Delta_{lm} = \frac{\sigma_{\oplus, c, s_{lm}} - \sigma_{c, s_{lm}}}{\sigma_{c, s_{lm}}} \]

Relative error of standard deviation

\[ \log_{10} |\Delta_{lm}| \].

<table>
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<tr>
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<th>min $\Delta_{lm}$</th>
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Nearly all differences negative, \( \sigma_{\oplus, c, s_{lm}} \) is always too optimistic!
Relative errors of standard deviations

\[ \Delta_{lm} = \frac{\sigma_{\oplus,c,s_{lm}} - \sigma_{c,s_{lm}}}{\sigma_{c,s_{lm}}} \]

Relative error of standard deviation

\[ \log_{10} |\Delta_{lm}|. \]
Relative errors of standard deviations

\[ \Delta_{lm} = \frac{\sigma_{\Theta, c, s_{lm}} - \sigma_{c, s_{lm}}}{\sigma_{c, s_{lm}}} \]

spherical harmonic degree \( l \)

\( \sin \leftrightarrow \) spherical harmonic order \( m \) \( \Rightarrow \) cos

\( \min = -0.301, \max = -0.000, \mean = -0.025, \rms = 0.035 \)

Relative error of standard deviation

\( \log_{10} |\Delta_{lm}|. \)

<table>
<thead>
<tr>
<th>ID</th>
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<td>+0.0</td>
<td>-0.23</td>
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<tr>
<td>5</td>
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<td>-0.0</td>
<td>-0.21</td>
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Nearly all differences negative, \( \sigma_{\Theta, c, s_{lm}} \) is always too optimistic!
Relative errors of standard deviations

\[ \Delta_{lm} = \frac{\sigma_{\pm,c,s_{lm}} - \sigma_{c,s_{lm}}}{\sigma_{c,s_{lm}}} \]

Relative error of standard deviation
\[ \log_{10}|\Delta_{lm}|. \]

<table>
<thead>
<tr>
<th>ID</th>
<th>min ( \Delta_{lm} ) %</th>
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Nearly all differences negative, \( \sigma_{\pm,c,s_{lm}} \) is always too optimistic!
Relative errors of standard deviations

\[ \Delta_{lm} = \frac{\sigma_{\oplus, c, s_{lm}} - \sigma_{c, s_{lm}}}{\sigma_{c, s_{lm}}} \]

\[ \sin \leftrightarrow \text{spherical harmonic order } m \Rightarrow \cos \]

\( \begin{array}{ccc}
\text{ID} & \Delta_{lm} & \Delta_{lm} & \Delta_{lm} \\
1 & -30.1 & -0.0 & -2.54 \\
2 & -1.7 & -0.0 & -0.23 \\
3 & -1.0 & +0.0 & -0.23 \\
5 & -1.0 & -0.0 & -0.21 \\
6 & -1.0 & -0.0 & -0.20 \\
\end{array} \]

Relative error of standard deviation

\[ \log_{10} |\Delta_{lm}|. \]

\[ \Rightarrow \text{Nearly all differences negative, } \sigma_{\oplus, c, s_{lm}} \text{ is always to optimistic!} \]
Propagate model errors to gravity field functionals: e.g. gravity anomalies

Reference:

\[ \Sigma_{g(\lambda,\varphi),g(\lambda,\varphi)} = FF^T = FN^{-1}F^T \]

Using the preconditioner/representative matrix:

\[ \Sigma_{\oplus,g(\lambda,\varphi),g(\lambda,\varphi)} = F\Sigma_{\oplus}F^T = FN_{\oplus}^{-1}F^T \]

avoiding fill-ins:

\[ \Sigma_{\oplus,g(\lambda,\varphi),g(\lambda,\varphi)} = F\left(R_{\oplus}^{-1}R_{\oplus}^{-1}\right)F^T = FR_{\oplus}^{-1}F^T \]

\[ = \bar{F}F^T \]
Reference errors on grid

**Reference:** Propagated errors, gravity anomalies \([m/s^2]\) using full GOCE_EGM_TIMRL0002 VCM

<table>
<thead>
<tr>
<th>Degree and order 2-250</th>
<th>Degree and order 2-200</th>
<th>Degree and order 2-100</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-12mgal</td>
<td>1-3mgal</td>
<td>0.1-0.25mgal</td>
</tr>
</tbody>
</table>

Use different preconditioners/representative matrices as model VCM and propagate to gravity anomaly error to evaluate quality.

**Note:** concentrate on preconditioners 2, 3, 5 and 6, as equatorial symmetry is obviously not the case.
propagated standard deviations $\sigma_{\oplus,g}$

<table>
<thead>
<tr>
<th>ID</th>
<th>$\min \Delta \sigma_g$ %</th>
<th>$\max \Delta \sigma_g$ %</th>
<th>$\text{mean} \Delta \sigma_g$ %</th>
<th>$\text{rms} \Delta \sigma_g$ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-6</td>
<td>-7.9</td>
<td>+3.3</td>
<td>-0.1</td>
<td>1.1</td>
</tr>
<tr>
<td>2-6</td>
<td>-3.0</td>
<td>+3.3</td>
<td>+0.0</td>
<td>0.7</td>
</tr>
</tbody>
</table>

relative error $|\Delta \sigma_g|$:

$\Delta \sigma_g = \frac{\sigma_{\oplus,g} - \sigma_g}{\sigma_g}$

Very similar behaviour for 4 preconditioners 2, 3, 5 and 6. Difference 0.1% digit!
Very similar behaviour for all 4 preconditioners 02, 04, 06 and 10. Difference 0.1% digit!

\[ \Delta \sigma_g = \frac{\sigma_{\oplus,g} - \sigma_g}{\sigma_g} \]
## Variance propagation d/o 2-100

### Propagated Standard Deviations $\sigma_{\oplus,g}$

![Map showing propagated standard deviations](image)

### Relative Error $|\Delta \sigma_g|$

![Map showing relative error](image)

| ID | $\min \Delta \sigma_g$ | $\max \Delta \sigma_g$ | $\mean \Delta \sigma_g$ | rms $\Delta \sigma_g$ |%
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-9.9</td>
<td>+2.3</td>
<td>-0.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>

$$\Delta \sigma_g = \frac{\sigma_{\oplus,g} - \sigma_g}{\sigma_g}$$

**2 bd + bd**

Brockmann and Schuh
Nürnberg, 27.09.2011
Preconditioners for GOCE
propagated standard deviations $\sigma_{\oplus,g}$

relative error $|\Delta \sigma_g|$ 

<table>
<thead>
<tr>
<th>ID</th>
<th>$\min \Delta \sigma_g$</th>
<th>$\max \Delta \sigma_g$</th>
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<th>$\text{rms} \Delta \sigma_g$</th>
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<tr>
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</tbody>
</table>

$\Delta \sigma_g = \frac{\sigma_{\oplus,g} - \sigma_g}{\sigma_g}$
Variance propagation d/o 2-100

propagated standard deviations $\sigma_{\oplus,g}$

relative error $|\Delta \sigma_g|$
propagated standard deviations $\sigma_{\oplus,g}$

<table>
<thead>
<tr>
<th>ID</th>
<th>min $\Delta \sigma_g$</th>
<th>max $\Delta \sigma_g$</th>
<th>mean $\Delta \sigma_g$</th>
<th>rms $\Delta \sigma_g$</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-9.9</td>
<td>+2.3</td>
<td>-0.1</td>
<td>1.1</td>
<td>global</td>
</tr>
<tr>
<td>3</td>
<td>-9.7</td>
<td>+2.1</td>
<td>-0.1</td>
<td>1.1</td>
<td>global</td>
</tr>
<tr>
<td>5</td>
<td>-0.9</td>
<td>+0.1</td>
<td>-0.0</td>
<td>0.1</td>
<td>global</td>
</tr>
<tr>
<td>6</td>
<td>-0.9</td>
<td>+0.1</td>
<td>-0.0</td>
<td>0.1</td>
<td>global</td>
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</tbody>
</table>

relative error $|\Delta \sigma_g|$ 

$\Delta \sigma_g = \frac{\sigma_{\oplus,g} - \sigma_g}{\sigma_g}$

6_freekite30
1 Motivation

2 Design of preconditioners for GOCE

3 Comparing different preconditioners
   - For CG acceleration
   - For estimation of spherical harmonics standard deviations
   - For variance propagation

4 Summary and Outlook
Summary and Conclusions

Summary:
- Tested preconditioners which do not assume equatorial symmetry improve condition in CG from $3 \cdot 10^6$ to 2.6
- Preconditioners (except simplest one) approximate all coefficient standard deviations at least to 1% (mean 0.2%)
- Tested preconditioners which do not assume equatorial symmetry are useful for variance propagation, error depending on degree ($\approx 3\%$ for d/o 250, $\approx 5\%$ for d/o 200, $\approx 1\%$ for d/o 100)
- Complex preconditioners contain most detail information (d/o 2-100)

Outlook:
- Alternative SST normal in combination
- Preconditioners for high resolution terrestrial data d/o 720+
Summary and Conclusions

2,500 × 2,500 submatrix of EGM_TIM_RL0002 gradiometer normal matrix (sorted by orders).

10,000 × 10,000 submatrix of EGM_TIM_RL0002 gradiometer correlation matrix (sorted by orders).

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