Anisotropic low-pass filters on the sphere

design & performance analysis

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In this presentation . . .

- *a primer on linear low-pass filtering on the sphere*
- *classification of filters, which leads to the different ways of designing filters*
- *performance analysis of filters, both objectively and subjectively*

understanding filtering and filters

Filtering on the sphere

$$\bar{f}(P) = \int_{\Omega'} b(P,Q) \quad f(Q) \quad \mathrm{d}\Omega'$$

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- we will only discuss linear low-pass filters.
- linear low-pass filtering is similar to weighted mean computation (weights are provided by $b(P,Q)$).
- filtering operations require two-points to be specified.

Filtering on the sphere

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$b(\cdot,\cdot)$ **is a two-point function**!

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where

$$f(\cdot) = \sum_{l,m} Y_{lm}(\cdot) K_{lm}$$

$$b(P,Q) = \sum_{l,m} Y_{lm}(P) \sum_{n,k} B_{lm}^{nk} Y_{nk}(Q)$$

Filtering: spatial-spectral duality

$$\bar{f}(P) = \int_{\Omega'} b(P,Q) \quad f(Q) \quad \mathrm{d}\Omega'$$

$$\bar{K}_{lm} = \sum_{n,k} B_{lm}^{nk} K_{nk}$$

Filters: from global frame to local frame

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$$b(P,Q) \quad \rightarrow \quad b(\theta,\lambda,\theta',\lambda')$$

Filters: from global frame to local frame

$$b(P, Q) \quad \rightarrow \quad b(\theta, \lambda, \psi, A)$$

Filters: from global frame to local frame

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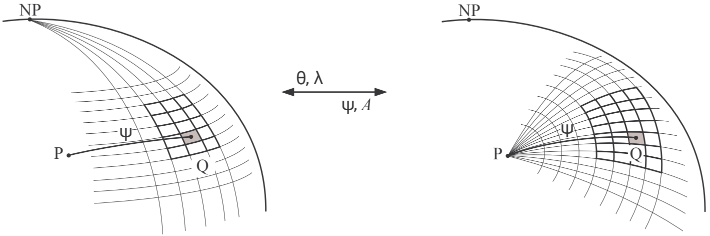

Classification of filters

Classification of filters

- Filter functions depend on
  - location of the filter $P\ (\theta, \lambda)$
  - spherical distance from filter location P to $Q\ (\psi)$
  - azimuth of Q w.r.t. $P\ (A)$

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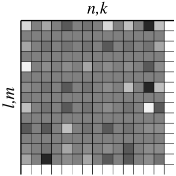
  - location of the filter $P\ (\theta, \lambda)$
  - spherical distance from filter location P to $Q\ (\psi)$
  - azimuth of Q w.r.t. $P\ (A)$
- Filter classification based on the three parameters
  - dependence on location (homogeneity)
  - dependence on direction (isotropy)

Classification of filters

	Inhomogeneous		Homogeneous
	(θ, λ)	(θ)	
Anisotropic			
Isotropic			

Classification of filters

	Inhomogeneous (θ, λ)	Inhomogeneous (θ)	Homogeneous
Anisotropic			
Isotropic			

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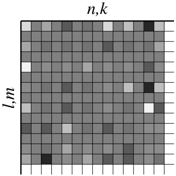
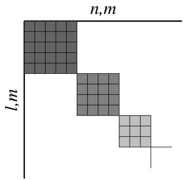
Anisotropic			
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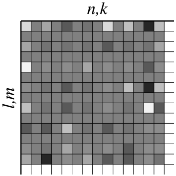
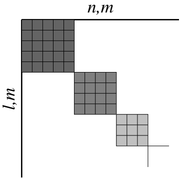
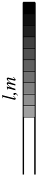
Anisotropic			
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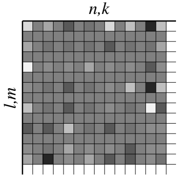
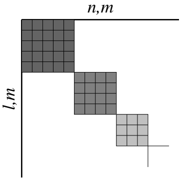
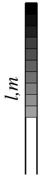
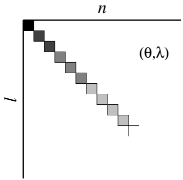
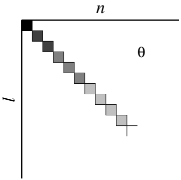
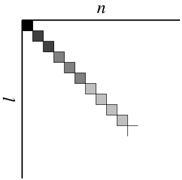
Anisotropic			
Isotropic			

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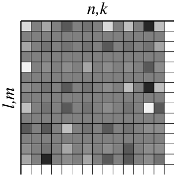
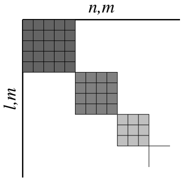
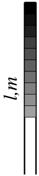
Anisotropic			

designing low-pass filter

Approaches to designing filters

Approaches to designing filters

Deterministic

Gaussian filter

$$b(\psi) \quad = \quad \zeta \exp\left\{-\beta \frac{1-\cos\psi}{1-\cos\psi_0}\right\}$$

Stochastic

Wiener filter

$$\mathbf{B} \quad = \quad \frac{\mathbf{S}}{\mathbf{S}+\mathbf{N}}$$

Data-driven

1. Principal Component Analysis
2. Independent Component Analysis
3. Canonical Correlation Analysis

analysing the performance of filters

FAQ about filters (in the GRACE community)

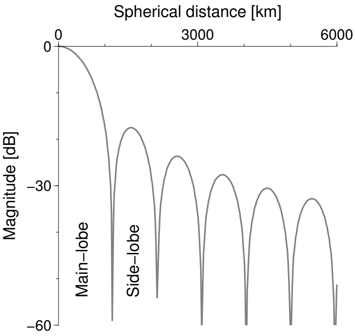
FAQ about filters (in the GRACE community)

- What is the signal loss due to filtering?
- How much signal do the filters "leak"?
- What is the resolution of filtered GRACE data?
- Is there a best filter for processing GRACE?

Basis for performance analysis

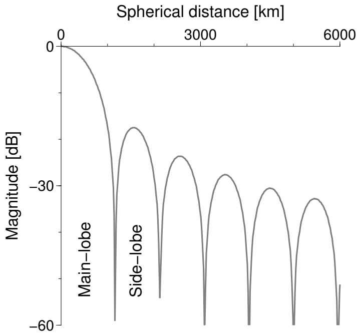
Basis for performance analysis

Filter anatomy



Basis for performance analysis

Filter anatomy



Energy function

Filter

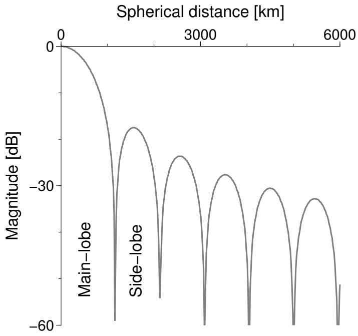
$$E(\theta,\lambda) \;=\; \int_{\Omega'} b^2(\theta,\lambda,\psi,A)\,\mathrm{d}\Omega'$$

Field

$$E_f \;=\; \int_{\Omega} f^2(\theta,\lambda)\,\mathrm{d}\Omega$$

Basis for performance analysis

Filter anatomy



Energy function

Filter

$$E(\theta,\lambda) = \int_{\Omega'} b^2(\theta,\lambda,\psi,A)\,\mathrm{d}\Omega'$$

Field

$$E_f = \sum_{l,m} K_{lm}^2$$

demo

Destriping filter + Gaussian 400 km filter

Destriping filter + Gaussian 400 km filter

Cascade filter

$$\bar{K}_{lm} = \sum_n B_{lm}^{nm} K_{nm}$$

Destriping filter + Gaussian 400 km filter

Cascade filter

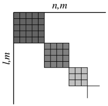
$$\bar{K}_{lm} = B_l \sum_n B_{lm}^{nm} K_{nm}$$

Destriping filter + Gaussian 400 km filter

Cascade filter

$$\bar{K}_{lm} = B_l \sum_n B_{lm}^{nm} K_{nm}$$

Block-diagonal spectrum



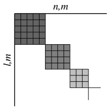
$$\rightarrow \quad b(\theta, \psi, A)$$

Destriping filter + Gaussian 400 km filter

Cascade filter

$$\bar{K}_{lm} = B_l \sum_n B_{lm}^{nm} K_{nm}$$

Block-diagonal spectrum



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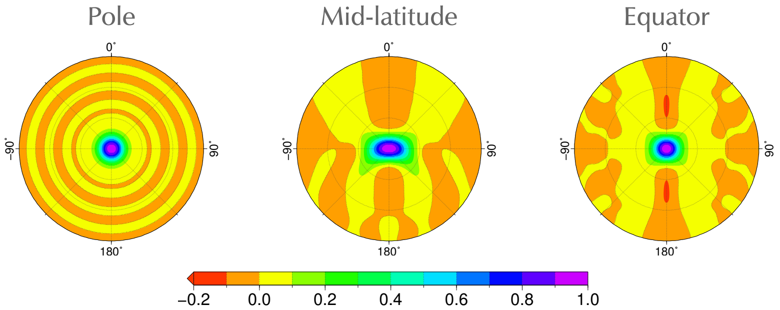


Signal lost due to filtering

Signal lost due to filtering

Damping factor

$$\alpha = \frac{E_{\bar{f}}}{E_f}$$

Signal lost due to filtering

Processing loss

$$\rho_{\mathrm{L}} = 1 - \alpha$$

Improving signal-to-noise ratio

Processing gain

$$\rho_G \quad = \quad \frac{\mathrm{SNR}_{\bar{f}}}{\mathrm{SNR}_f}$$

Global measures

Destriping + Gaussian 400 km

Damping factor (α)	0.52
Processing loss (ρ_L)	−6 dB
Processing gain (ρ_G)	62 dB

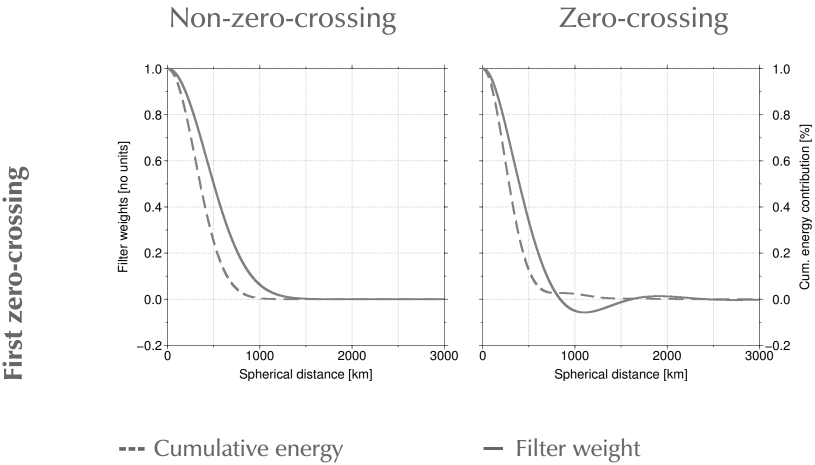
Signal leakage due to filtering

Signal leakage due to filtering

Leakage – signal contribution from outside the *filter width*

Signal leakage due to filtering

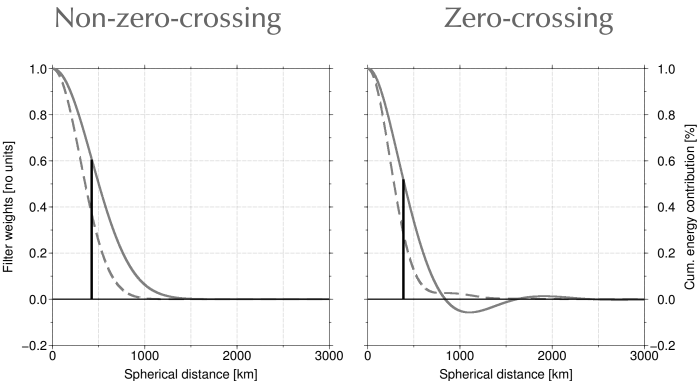
Filter width



Signal leakage due to filtering

Filter width

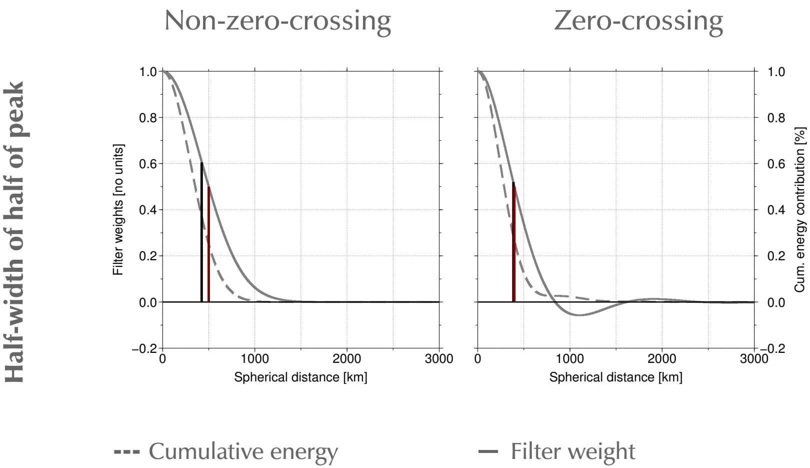
Spatial variance of energy



--- Cumulative energy — Filter weight

Signal leakage due to filtering

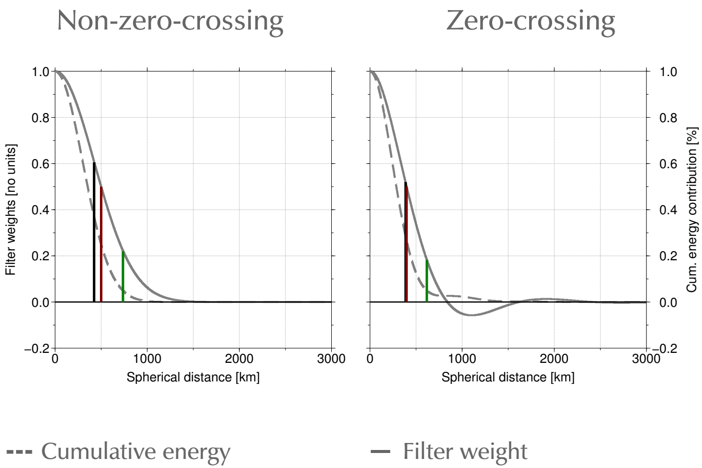
Filter width



Signal leakage due to filtering

Filter width

95% of total filter energy



Signal leakage due to filtering

Filter width → spatial variance

$$\psi_c^2(\theta,\lambda) = \int_{\Omega'} \psi^2 \, \frac{b^2(\theta,\lambda,\psi,A)}{E(\theta,\lambda)} \, \mathrm{d}\Omega'$$

Signal leakage due to filtering

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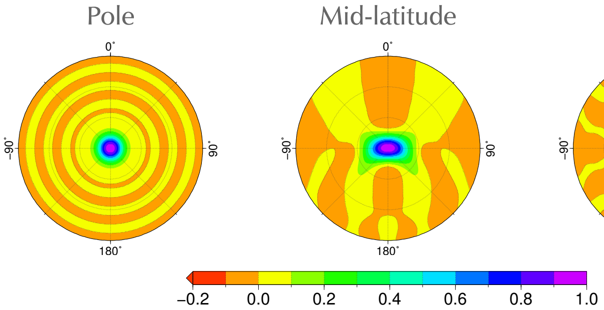

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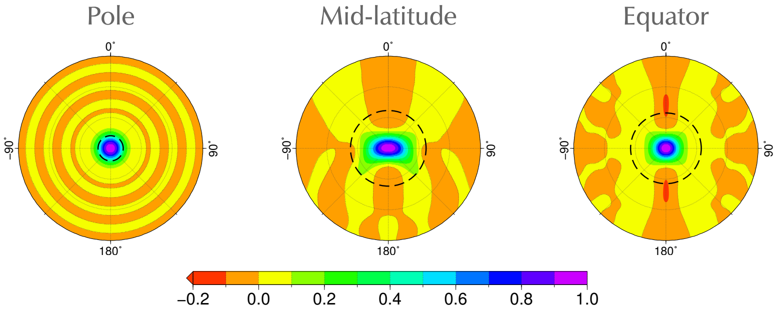
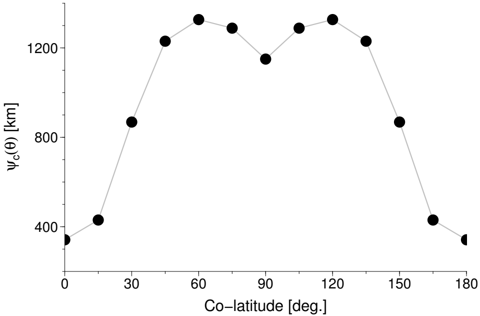


Signal leakage due to filtering

Spatial variance



Signal leakage due to filtering

Spatial leakage

$$\xi(\theta,\lambda) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\psi_c}^{\pi}\frac{b^2(\theta,\lambda,\psi,A)}{E(\theta,\lambda)}\sin\psi\,d\psi\,dA$$

Signal leakage due to filtering

Spatial leakage

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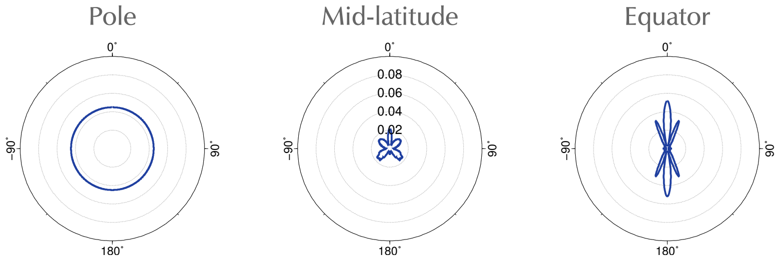
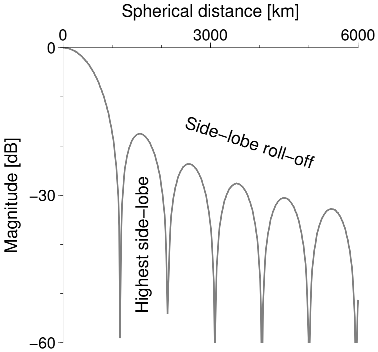


Signal leakage due to filtering

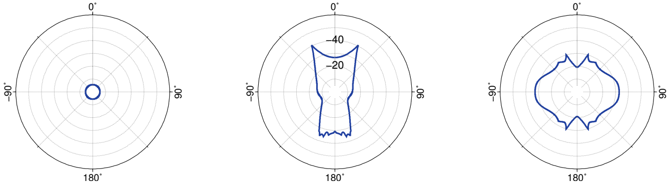
the real culprits



Signal leakage due to filtering

Pole | Mid-latitude | Equator

Highest side-lobe



Signal leakage due to filtering

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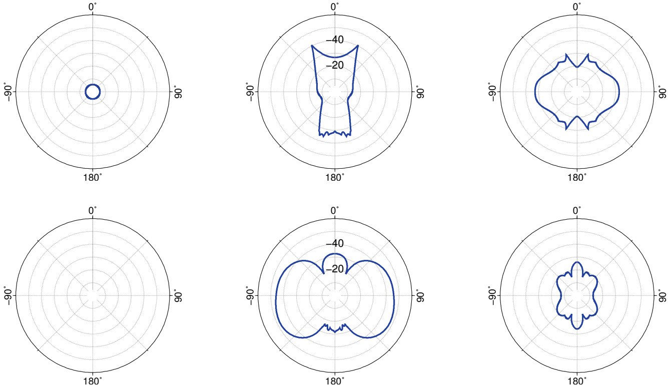
Roll-off



Finding the resolution of filters and the "best" filter are left as an excercise for the audience

Summary

Anisotropic linear low-pass filtering on the sphere

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Classification

- Parameters – (θ, λ), ψ, A
- Poperties – Isotropy and homogeneity

Design

1. Deterministic
2. Stochastic
3. Data-driven

Anisotropic linear low-pass filtering on the sphere

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Global performance

1. Processing gain
2. Processing loss

Local performance

1. Spatial leakage
2. Spatial variance
3. Highest side-lobe
4. Side-lobe roll-off

Thank you!