Cubature Particle filter applied in a tightly-coupled GPS/INS navigation system

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Main Contents

I. The Cubature Particle Filter (CPF)
   - Gaussian Particle Filter (GPF)
   - Cubature Kalman Filter (CKF)

II. GPS/INS tightly-coupled navigation system
   - Nonlinear IMU model
   - GPS/INS measurement model

III. Navigation results
   - Experiment
   - Comparison

IV. Conclusion & Future work
   - Conclusion
   - Future work
The Cubature Particle Filter

- The Gaussian Particle Filter
  - Use the Gaussian distribution to approximate the importance density function in the Particle Filter
  - No particles’ degeneration
  - No need in resampling the particles
  - Reduce the computational burden

- The Cubature Kalman Filter
  - The heart is the spherical-radial cubature rules
  - Use 2n equal weighted cubature points
  - A third-order approximation to the nonlinear system

- The Cubature Particle Filter
  - Use the state estimated by the CKF to form the importance density function
The Cubature Kalman Filter

- **Time update**

\[
\begin{align*}
P_{k|k-1} &= S_{k-1|k-1} S_{k-1|k-1}^T \\
\chi_{k-1|k-1} &= S_{k-1|k-1} \xi + x_{k-1|k-1} \\
\chi^*_{k-1|k-1} &= f(\chi_{k-1|k-1}, u_{k-1}) \\
x_{k|k-1} &= \frac{1}{m} \sum_{i=1}^{m} \chi^*_{i, k-1|k-1} \\
P_{k|k-1} &= \frac{1}{m} \sum_{i=1}^{m} \chi^*_{i, k|k-1} \chi^*_{i, k|k-1}^T - x_{k|k-1}^T x_{k|k-1} + Q_{k-1}
\end{align*}
\]

- **Measurement update**

\[
\begin{align*}
P_{k|k-1} &= S_{k|k-1} S_{k|k-1}^T \\
\chi_{k|k-1} &= S_{k|k-1} \xi + x_{k|k-1} \\
Z_{k-1|k} &= h(\chi_{k-1|k-1}, u_{k-1}) \\
z_{k|k-1} &= \frac{1}{m} \sum_{i=1}^{m} Z_{i, k-1|k-1} \\
P_{zz, k|k-1} &= \frac{1}{m} \sum_{i=1}^{m} Z_{i, k|k-1} Z_{i, k|k-1}^T - z_{k|k-1} z_{k|k-1}^T + R_k \\
P_{xz, k|k-1} &= \frac{1}{m} \sum_{i=1}^{m} \chi_{i, k|k-1} Z_{i, k|k-1}^T - x_{k|k-1} z_{k|k-1}^T \\
K_k &= P_{xz, k|k-1} P_{zz, k|k-1}^{-1} \\
x_{k|k} &= x_{k|k-1} + K_k (z_k - z_{k|k-1}) \\
P_{k|k} &= P_{k|k-1} - K_k P_{zz, k|k-1} K_k^T
\end{align*}
\]
The Cubature Particle Filter

- The particles will be generated using the CKF a posteriori estimates as
  \[ X_{k,i} \sim N(x_{k|k}, P_{k|k}) \]
- The weights can be calculated as
  \[
  \omega(X_{k,i}) = \frac{p(z_k \mid X_{k,i})N(X_{k,i} \mid x_{k|k-1}, P_{k|k-1})}{N(X_{k,i} \mid x_{k|k}, P_{k|k})} \\
  \tilde{\omega}(X_{k,i}) = \frac{\omega(X_{k,i})}{\sum_{i=1}^{N} \omega(X_{k,i})}
  \]
- The states and covariance can be estimated as
  \[
  x_{k}^{+} = \sum_{i=1}^{N} \tilde{\omega}(X_{k,i})X_{k,i} \\
  P_{k}^{+} = \sum_{i=1}^{N} \tilde{\omega}(X_{k,i})[X_{k,i} - x_{k}^{+}][X_{k,i} - x_{k}^{+}]^T
  \]
The Cubature Particle Filter

Initialization

Cubature Kalman Filter

Calculate the importance weights

Get the new estimates

\[ (x_{k|k-1}^+, P_{k|k-1}^+) \]

\[ (z, R) \]

\[ X_{k,i} \overset{\sim}{\sim} N(x_{k|k}, P_{k|k}), \omega_{k,i} = 1/N \]
GPS/INS tightly-coupled navigation system

- System structure

![Diagram of GPS/INS tightly-coupled navigation system]

- IMU
- GPS
- Mechanization
- Filter
- Ephemeris
- Pseudorange
- Pseudorange rate
- Pseudorange Doppler
- P, V, A
- IMU Errors
- Angular velocity
- Acceleration
GPS/INS tightly-coupled navigation system

- State vector

\[ x = [\delta \alpha, \delta \beta, \delta \gamma, \delta \nu_x, \delta \nu_y, \delta \nu_z, \delta l, \delta b, \delta h, \nabla_x, \nabla_y, \nabla_z, \varepsilon_x, \varepsilon_y, \varepsilon_z, c\delta t, c\dot{\delta}t] \]

- The difference between the true value and estimated value
- \( \delta \alpha, \delta \beta, \delta \gamma \): the attitude error
- \( \delta \nu_x, \delta \nu_y, \delta \nu_z \): the velocity error
- \( \delta l, \delta b, \delta h \): the position error (longitude, latitude and height)
- \( \nabla_x, \nabla_y, \nabla_z \): the gyro drift
- \( \varepsilon_x, \varepsilon_y, \varepsilon_z \): the acceleration bias
- \( c\delta t \): the receiver clock offset expressed in meters
- \( c\dot{\delta}t \): the receiver clock drift expressed in meters
GPS/INS tightly-coupled navigation system

- Transition function

\[
\begin{align*}
\dot{\phi} &= (I - C_n^\tilde{n})\tilde{\omega}_b^n + C_n^\tilde{n}\delta\omega^n - C_b^\tilde{n}\delta\omega^b \\
\delta\dot{v}^n &= [I - C_n^\tilde{n}]\tilde{C}_b^\tilde{n}\tilde{f}^b + C_b^\tilde{n}\delta f^b - (2\delta\omega_{ie}^n + \delta\omega_{en}^n) \times v^n - (2\tilde{\omega}_{ie}^n + \tilde{\omega}_{en}^n) \times \delta v^n + \delta g^n \\
\delta\dot{L} &= \frac{\delta v^n_N}{M + h} - \frac{\delta hv^n_N}{(M + h)^2} \\
\delta\dot{b} &= \frac{\delta v^n_E \sec L}{N + h} + \frac{\delta L v^n_E \tan L \sec L}{N + h} - \frac{\delta hv^n_E \sec L}{(N + h)^2} \\
\delta\dot{h} &= \delta v^n_U \\
\dot{\nu} &= -\beta\nu \\
\dot{\epsilon} &= -\beta\epsilon
\end{align*}
\]
GPS/INS tightly-coupled navigation system

- An approximation to the attitude error
  - The attitude error is treated as the rotational angle between the true and estimated navigation frames
- The direction cosine matrix can be used
- Different from the psi-angle expression

\[
C^n = \begin{bmatrix}
\cos \delta \beta \cos \delta \gamma - \sin \delta \beta \sin \delta \alpha \sin \delta \gamma & \cos \delta \beta \sin \delta \gamma + \sin \delta \beta \sin \delta \alpha \cos \delta \gamma & -\sin \delta \beta \cos \delta \alpha \\
-\cos \delta \alpha \sin \delta \gamma & \cos \delta \alpha \cos \delta \gamma & \sin \delta \alpha \\
\sin \delta \beta \cos \delta \gamma + \cos \delta \beta \sin \delta \alpha \sin \delta \gamma & \sin \delta \beta \sin \delta \gamma - \cos \delta \beta \sin \delta \alpha \cos \delta \gamma & \cos \delta \beta \cos \delta \alpha
\end{bmatrix}
\]
GPS/INS tightly-coupled navigation system

- Measurement update
  - Pseudo-range and Pseudo-range rate
  - Difference between the observation and the estimated value
  - Position is transferred from ECEF to LBH

Pseudo-range

\[ r_j = \sqrt{(x_{s,j} - x)^2 + (y_{s,j} - y)^2 + (z_{s,j} - z)^2} \]

Difference

\[ \delta \rho = c \delta t_r - e_{j1} \delta x - e_{j2} \delta y - e_{j3} \delta z \]

Pseudo-range rate

\[ \dot{r}_j = e_{j1} (\dot{x}_{s,j} - \dot{x}) + e_{j1} (\dot{y}_{s,j} - \dot{y}) + e_{j1} (\dot{z}_{s,j} - \dot{z}) \]

Difference

\[ \delta \dot{\rho}_j = g_{j1} \delta x + g_{j2} \delta y + g_{j3} \delta z + e_{j1} \delta \dot{x} + e_{j2} \delta \dot{y} + e_{j3} \delta \dot{z} + c \delta t_r \]
## Experiment

### MEMS IMU

<table>
<thead>
<tr>
<th></th>
<th>Gyroscope</th>
<th>Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias</td>
<td>0.1 °/s</td>
<td>0.1 mg</td>
</tr>
<tr>
<td>Noise</td>
<td>2 °/√h</td>
<td>36 μg/√Hz</td>
</tr>
</tbody>
</table>

### RLG IMU

<table>
<thead>
<tr>
<th></th>
<th>Gyroscope</th>
<th>Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias</td>
<td>0.003 °/h</td>
<td>25 μg</td>
</tr>
<tr>
<td>Noise</td>
<td>0.002 °/√h</td>
<td>8 μg/√Hz</td>
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</tbody>
</table>
Experiment
Comparison among different particle’s number--Attitude
## Comparison among different particle’s number--Attitude

<table>
<thead>
<tr>
<th></th>
<th>Roll(deg)</th>
<th></th>
<th>Pitch(deg)</th>
<th></th>
<th>Yaw(deg)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Max</td>
<td>Mean</td>
<td>RMS</td>
<td>Max</td>
<td>Mean</td>
<td>RMS</td>
</tr>
<tr>
<td>100</td>
<td>5.0113</td>
<td>0.6303</td>
<td>0.7166</td>
<td>9.3864</td>
<td>0.5758</td>
<td>0.8762</td>
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<tr>
<td>200</td>
<td>4.8550</td>
<td>0.2589</td>
<td>0.4959</td>
<td>8.5296</td>
<td>0.2792</td>
<td>0.7031</td>
</tr>
<tr>
<td>500</td>
<td>3.6620</td>
<td>0.2578</td>
<td>0.3785</td>
<td>5.8755</td>
<td>0.2281</td>
<td>0.4332</td>
</tr>
<tr>
<td>1000</td>
<td>3.8888</td>
<td>0.2119</td>
<td>0.3798</td>
<td>4.9648</td>
<td>0.2392</td>
<td>0.4191</td>
</tr>
<tr>
<td>2000</td>
<td>3.3746</td>
<td>0.3456</td>
<td>0.4069</td>
<td>4.8089</td>
<td>0.2358</td>
<td>0.4259</td>
</tr>
</tbody>
</table>
Comparison among different filtering methods--Attitude

- Roll (deg) vs. UTC time (s)
- Pitch (deg) vs. UTC time (s)
- Yaw (deg) vs. UTC time (s)
## Comparison among different filtering methods--Attitude

<table>
<thead>
<tr>
<th></th>
<th>Roll(deg)</th>
<th></th>
<th>Pitch(deg)</th>
<th></th>
<th>Yaw(deg)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Max</td>
<td>Mean</td>
<td>RMS</td>
<td>Max</td>
<td>Mean</td>
<td>RMS</td>
</tr>
<tr>
<td>EKF</td>
<td>3.7100</td>
<td>0.3054</td>
<td>0.3987</td>
<td>4.8508</td>
<td>0.2598</td>
<td>0.4511</td>
</tr>
<tr>
<td>CKF</td>
<td>3.4186</td>
<td>0.2214</td>
<td>0.3890</td>
<td>5.2898</td>
<td>0.2088</td>
<td>0.4047</td>
</tr>
<tr>
<td>CPF</td>
<td>3.8888</td>
<td>0.2119</td>
<td>0.3798</td>
<td>4.9648</td>
<td>0.2392</td>
<td>0.4191</td>
</tr>
</tbody>
</table>

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17 | Yingwei Zhao & David Becker | InterGeo | 08/10/13
Coasting Performance
## Coasting-Maximum Position difference

<table>
<thead>
<tr>
<th></th>
<th>Easting(m)</th>
<th>Northing(m)</th>
<th>Height(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CT 1</td>
<td>CT 2</td>
<td>CT 3</td>
</tr>
<tr>
<td>EKF</td>
<td>34.434</td>
<td>105.63</td>
<td>335.90</td>
</tr>
<tr>
<td>CKF</td>
<td>39.984</td>
<td>1.3947</td>
<td>35.570</td>
</tr>
</tbody>
</table>
Conclusion & Future work

**Conclusion**
- The CPF performs better than the EKF
- The CPF shows similar performance with the CKF using Gaussian distribution
- The CKF can ease the curse of dimensionality, but can’t eliminate it
- High computation burden

**Future work**
- Rao-Blackwellized Cubature Particle Filter
- Gaussian-sum Cubature Particle Filter
Thank you very much for the attention!!

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Comparison between the UKF and the CKF

- **Difference**
  - CKF: 2n Cubature points
  - UKF: 2n+1 Sigma points
  - CKF: all the weights are positive, no possibility in negative definite
  - UKF: has the possibility in negative definite
  - CKF: more suitable for higher-order system
  - UKF: more suitable for lower-order system
  - CKF: proved in theory
  - UKF: based on the assumption
  - CKF: only has one parameter to tune
  - UKF: has three parameters to tune

- **Similarity**
  - A third-order approximation to the nonlinear system
  - The CKF can be treated as a special case of the UKF
  - Gaussian filter
The function of the CKF in the CPF

Bootstrap Particle Filter
The function of the CKF in the CPF

Cubature Particle Filter
The reason of the jumps in the beginning stage in the attitude
## The observability of the attitude error

<table>
<thead>
<tr>
<th></th>
<th>$a_E=0$</th>
<th>$a_E\neq0$</th>
<th>S maneuvering</th>
<th>Turning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta\alpha$</td>
<td>$1.2e^{-7}$</td>
<td>$1.2e^{-7}$</td>
<td>$1.7e^{-7}$</td>
<td>$1.3e^{-7}$</td>
</tr>
<tr>
<td>$\delta\beta$</td>
<td>$1.2e^{-7}$</td>
<td>$1.2e^{-7}$</td>
<td>$1.7e^{-7}$</td>
<td>$1.3e^{-7}$</td>
</tr>
<tr>
<td>$\delta\gamma$</td>
<td>$1.4e^{-3}$</td>
<td>$9.3e^{-4}$</td>
<td>$8.5e^{-4}$</td>
<td>$2.8e^{-5}$</td>
</tr>
</tbody>
</table>