Construction of directional wavelets on the sphere

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Motivation

- model refinement by localizing base functions
  - tend to zero outside the area of influence
  - model mainly data within the area of interest

Fig: Mascons (Lemoine, 2007)

Fig: Boundary elements (Weigelt, 2012)
Motivation

model refinement by localizing base functions

- tend to zero outside the area of influence
- model mainly data within the area of interest
- very often: radial symmetric

Fig: Radial base functions on a sphere

Fig: Mascons (Lemoine, 2007)

Fig: Boundary elements (Weigelt, 2012)
Motivation

- observations by satellites have
  - preferred direction
  - converging of tracks

Fig: Spherical caps with radius 1.5° and points per cap
Idea

- isotropic functions $\psi(x, y)$ in spatial domain
  ($x$: location, $y$: node/center)
- linear transformation $\tilde{x} = E \cdot x$ and $\tilde{y} = E \cdot y$
- ‘elliptical’ contour lines per wavelets
Poisson wavelets

Poisson wavelets of order $N$:

$$\chi_n = \left( \frac{\|y\|}{\|y\|} \frac{\partial}{\partial \|y\|} \right)^n \frac{1}{\|x - y\|}$$

for $n = 0, 1, \ldots, N + 1$ and

$$\psi(x, y) = \frac{1}{4\pi R^2} \left( 2\chi_{N+1} + \chi_N \right)$$

recursive formulas up to $N = 9$

Fig: (Normalized) wavelet on the sphere

Fig: Cut along the meridian
Transformation

- keep size in North-South direction
- scaling in East-West direction
- empirical factor: $f(\phi) := \exp\left(\frac{1}{2} - \frac{1}{2} \left(\frac{\phi}{45}\right)^2\right)$

Fig: Points within modified spherical caps and points per cap
‘Elliptical’ wavelets

\[
E^{-1} = (R^e_g)^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} R^e_g \quad \text{with} \quad R^e_g = R_2(90 - \phi)R_3(\lambda)
\]

\[\Rightarrow \psi(E_x, E_y)\]

Fig: (Normalized) original and modified wavelet on the sphere
Spherical grid

Fig: Fibonacci grid (depth = 100 km)

- well suited for standard wavelets
- not enough nodes for ‘elliptic’ wavelets
Spherical grid

Fig: Fibonacci grid (depth = 100 km)

Fig: ‘scaled helix grid’
Example

- **GRACE-like orbit parameter**
- energy balance approach
- subtraction of a reference field
- ‘regional’ selection
- statistic in \([m^2/s^2]\):

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Value</th>
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<tbody>
<tr>
<td>MEAN</td>
<td>0.1191</td>
</tr>
<tr>
<td>MAX</td>
<td>0.9814</td>
</tr>
<tr>
<td>MIN</td>
<td>−0.5464</td>
</tr>
<tr>
<td>STD</td>
<td>0.2774</td>
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</tbody>
</table>

Fig: Potential in space
### Synthesis

**Fig:** Synthesis by radial and ‘elliptic’ wavelets

<table>
<thead>
<tr>
<th></th>
<th>radial</th>
<th>‘elliptic’</th>
</tr>
</thead>
<tbody>
<tr>
<td>nodes</td>
<td>623</td>
<td>598</td>
</tr>
<tr>
<td>$\text{cond} \left( A^T A \right)$</td>
<td>1709</td>
<td>495</td>
</tr>
<tr>
<td>regularization</td>
<td>$3.58 \cdot 10^{-7}$</td>
<td>$2.56 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>correlation [%]</td>
<td>0.94 (0.99)</td>
<td>0.93 (0.99)</td>
</tr>
</tbody>
</table>
Residuals in the inner zone

Fig: residuals after wavelet synthesis

<table>
<thead>
<tr>
<th>$[\text{m}^2/\text{s}^2]$</th>
<th>synthesis</th>
<th>residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>radial</td>
<td>‘elliptic’</td>
</tr>
<tr>
<td>MAX</td>
<td>1.0101</td>
<td>1.0325</td>
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<tr>
<td>MIN</td>
<td>−0.6260</td>
<td>−0.6138</td>
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<td>MEAN</td>
<td>0.1137</td>
<td>0.1095</td>
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<tr>
<td>STD</td>
<td>0.2782</td>
<td>0.2769</td>
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</table>
Comparison

So far, similar quality for both kinds of wavelets

But for ‘elliptic’ wavelets
- smaller condition number and regularization parameter
- consideration of observation geometry
- improvements by grid modifications
Summary

construction of spherical base functions
- non-isotropic/directional dependent
- re-scaling in East-West direction with latitude
- in spatial domain
- easy/fast calculation
Summary

- construction of spherical base functions
  - non-isotropic/directional dependent
  - re-scaling in East-West direction with latitude
  - in spatial domain
  - easy/fast calculation

Open Questions

- scaling should depend on orbital parameters \((l,r)\)
- unchanged caps in higher/lower latitudes?
- analysis in other directions?
- special grid design?