Pre-mission error assessment for the pendulum formation via the semi-analytical approach

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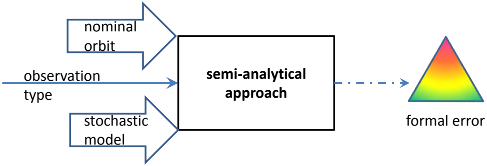


Fig: scheme of the semi-analytical approach

+ very effective error assessment
+ no observation data
– constant orbital parameters
– constant gravity field
– representation in lumped coefficients required

Semi-analytical approach

1. rotating Hill-frame
2. lumped coefficient = 2D Fourier series

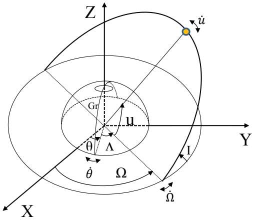
$$f^{\#} = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} A_{mk}^{\#} e^{\imath(ku+m\Lambda)}$$

$$A_{mk}^{\#} = \sum_{l=\max(|m|,|k|)}^{\infty} H_{lmk}^{\#} \bar{K}_{lm}$$

$f^{\#}$: specific observable

$H_{lmk}^{\#}$: transfer coefficients

$\bar{K}_{lm}$: gravity field coefficients



Semi-analytical approach

1. rotating Hill-frame
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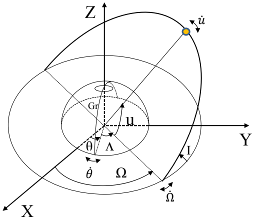
$$f^{\#} = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} A^{\#}_{mk} e^{\imath(ku+m\Lambda)} = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} A^{\#}_{mk} e^{\imath\dot{\psi}_{km}t}$$

$$A^{\#}_{mk} = \sum_{l=\max(|m|,|k|)}^{\infty} H^{\#}_{lmk} \bar{K}_{lm}$$



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circular case: $\dot{\psi}_{km} = (k\dot{u} + m\dot{\Lambda})$

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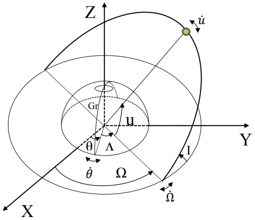

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3. error propagation in matrix form:

$$\underline{\mathbf{Q}}_{xx} = \left[\left(\underline{\mathbf{H}}^{\#}_{lmk} \right)^{\top} \underline{\mathbf{P}} \left(\underline{\mathbf{H}}^{\#}_{lmk} \right) \right]^{-1} \Rightarrow \delta \bar{K}_{lm} \Rightarrow \{\delta \bar{C}_{lm}, \delta \bar{S}_{lm}\}$$

Hill differential equation

orbit perturbations in rotating frame:

$$\ddot{x} + 2n\dot{z} = \sum_{l,m,k} \frac{GM}{R^2} \left(\frac{R}{r}\right)^{l+2} \imath k \bar{F}_{lmk}(I) \bar{K}_{lm} \, e^{\imath \dot{\psi}_{km} t}$$

$$\ddot{y} + n^2 y = \sum_{l,m,k} \frac{GM}{R^2} \left(\frac{R}{r}\right)^{l+2} \bar{F}^*_{lmk}(I) \bar{K}_{lm} \, e^{\imath \dot{\psi}_{km} t}$$

$$\ddot{z} - 2n\dot{x} - 3n^2 z = \sum_{l,m,k} \frac{GM}{R^2} \left(\frac{R}{r}\right)^{l+2} (-l-1) k \bar{F}_{lmk}(I) \bar{K}_{lm} \, e^{\imath \dot{\psi}_{km} t}$$

- n: natural orbit frequency ($n^2 r^3 = GM$)
- disturbing force = gravity gradient in rotation frame
- (cross-track) **inclination function** $\bar{F}_{lmk}(I)$ ($\bar{F}^*_{lmk}(I)$)

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Transfer coefficients of orbit perturbations

Transfer coefficients of particular solution:

$$H_{lmk}^{\Delta x} = \frac{2(l+1)\dot{\psi}_{mk} n - k(\dot{\psi}_{mk}^2 + 3n^2)}{\dot{\psi}_{mk}^2(\dot{\psi}_{mk}^2 - n^2)} \imath \frac{GM}{R^2} \left(\frac{R}{r}\right)^{l+2} \bar{F}_{lmk}(I)$$

$$H_{lmk}^{\Delta y} = \frac{1}{n^2 - \dot{\psi}_{mk}^2} \frac{GM}{R^2} \left(\frac{R}{r}\right)^{l+2} \bar{F}_{lmk}^{*}(I)$$

$$H_{lmk}^{\Delta z} = \frac{(l+1)\dot{\psi}_{mk} - 2kn}{\dot{\psi}_{mk}^2(\dot{\psi}_{mk}^2 - n^2)} \frac{GM}{R^2} \left(\frac{R}{r}\right)^{l+2} \bar{F}_{lmk}(I)$$

orbit perturbations

$$\Delta x(t) = \sum_{l,m,k} H_{lmk}^{\Delta x} \bar{K}_{lm} \, \mathrm{e}^{\imath \dot{\psi}_{km} t}$$

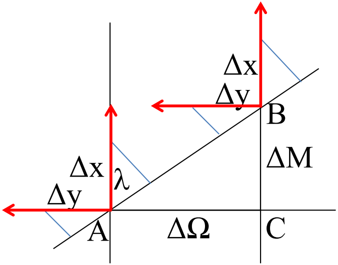
$$\ldots$$

Pendulum constellation

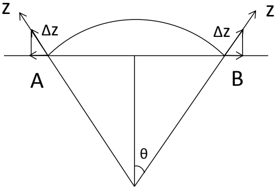
- 2 satellites on circular orbits
- same inclination I and radius r
- different ascending node $\Delta\Omega$
- 'time lag' ΔM between satellites

- observation: range, range rate, range acceleration
- time dependent opening angle
  ($\Rightarrow$ along-track and cross-track observations)

Range rate pertubation



(a) along/cross track directions



(b) radial direction

planar approximation

$$\Delta\rho(t) = (\Delta x_B - \Delta x_A)\cos\lambda + (\Delta y_A - \Delta y_B)\sin\lambda + (\Delta z_B + \Delta z_A)\sin\theta$$

Orit perturbations

$$\Delta\rho(t) = (\Delta x_{\mathrm{B}} - \Delta x_{\mathrm{A}}) \cos\lambda + (\Delta y_{\mathrm{A}} - \Delta y_{\mathrm{B}}) \sin\lambda + (\Delta z_{\mathrm{B}} + \Delta z_{\mathrm{A}}) \sin\theta$$

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Orit perturbations

$$\Delta\rho(t) = (\Delta x_B - \Delta x_A)\cos\lambda + (\Delta y_A - \Delta y_B)\sin\lambda + (\Delta z_B + \Delta z_A)\sin\theta$$

components:

$$\Delta x_B - \Delta x_A = \sum_{l,m,k} H_{lmk}^{\Delta x}\left(e^{\imath(\beta_{mk}\Delta M + m\Delta\Omega)} - 1\right)\bar{K}_{lm}\ e^{\imath\dot{\psi}_{km}t}$$

$$\Delta y_A - \Delta y_B = \sum_{l,m,k} H_{lmk}^{\Delta y}\left(1 - e^{\imath(\beta_{mk}\Delta M + m\Delta\Omega)}\right)\bar{K}_{lm}\ e^{\imath\dot{\psi}_{km}t}$$

$$\Delta z_A + \Delta z_B = \sum_{l,m,k} H_{lmk}^{\Delta z}\left(e^{\imath(\beta_{mk}\Delta M + m\Delta\Omega)} + 1\right)\bar{K}_{lm}\ e^{\imath\dot{\psi}_{km}t}$$

with $\beta_{mk} = k + m\frac{\dot{\Lambda}}{\dot{u}}$

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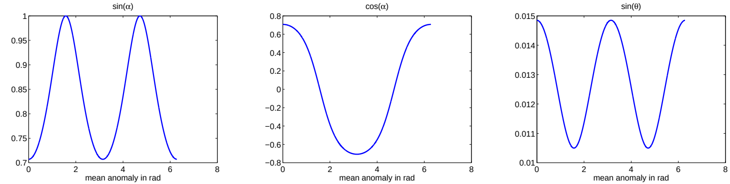
Time-variable orbital elements



(a) $\cos\lambda$ (b) $\sin\lambda$ (c) $\sin\theta$

Fig: time variable components for $\rho_x = \rho_{y,max} = 141\,\mathrm{km}$ ($\lambda = 45^\circ$)

$$\cos\lambda = \sum_{k'=-K}^{K} A_{k'} \mathrm{e}^{ik'\dot{u}t}, \quad \sin\lambda = \sum_{k'=-K}^{K} B_{k'} \mathrm{e}^{ik'\dot{u}t},$$

$$\sin\theta = \sum_{k'=-K}^{K} C_{k'} \mathrm{e}^{ik'\dot{u}t}$$

- numerical FFT
- recursion formula (elliptic integrals)

Transfer coefficients of the pendulum formation

$$H_{lmp}^{\Delta\rho(t)} = H_{lmp}^{\Delta\rho(t)_x} + H_{lmp}^{\Delta\rho(t)_y} + H_{lmp}^{\Delta\rho(t)_z} \quad \text{with}$$

$$H_{lmp}^{\Delta\rho(t)_x} = \sum_{k'=-K}^{K} A_{k'} H_{lm(p-k')}^{\Delta x} \left(\mathrm{e}^{\imath(\beta_{m(p-k')}\Delta M + m\Delta\Omega)} - 1 \right)$$

$$H_{lmp}^{\Delta\rho(t)_y} = \sum_{k'=-K}^{K} B_{k'} H_{lm(p-k')}^{\Delta x} \left(1 - \mathrm{e}^{\imath(\beta_{m(p-k')}\Delta M + m\Delta\Omega)} \right)$$

$$H_{lmp}^{\Delta\rho(t)_z} = \sum_{k'=-K}^{K} C_{k'} H_{lm(p-k')}^{\Delta x} \left(\mathrm{e}^{\imath(\beta_{m(p-k')}\Delta M + m\Delta\Omega)} + 1 \right)$$

new index $p = k + k'$

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Effect of Fourier series

terms like $\sum_{k'=-K}^{K} A_{k'} H^{\Delta x}_{lm(p-k')}$ describe a convolution

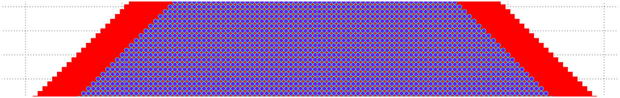
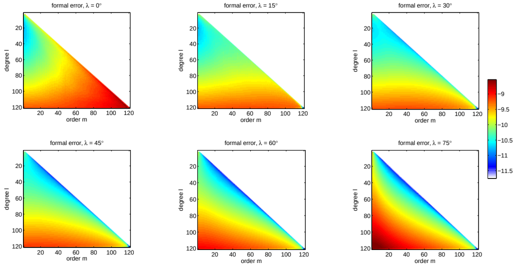


Fig: non-zero elements of transfer matrix for degree $l = 60$, $m = 20$, $K = 9$ (blue: GRACE, red+blue: pendulum))

Formal error for range rate



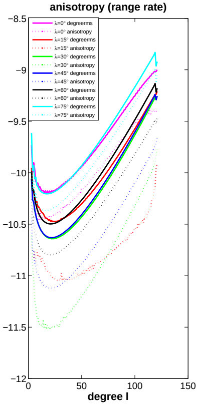
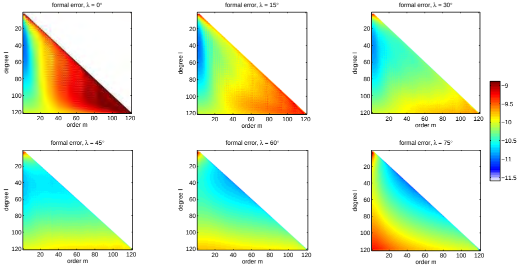


Fig: $\beta/\alpha = 490/31$, $\rho_{max} = 200\,\text{km}$, $h = 308\,\text{km}$ and $psd = 5 \cdot 10^{-5}\,\mathrm{m/s^2/\sqrt{Hz}}$

- most homogeneous error pattern for $\lambda \approx 30°$
- smallest formal errors $30° - 45°$

Formal error for range acceleration



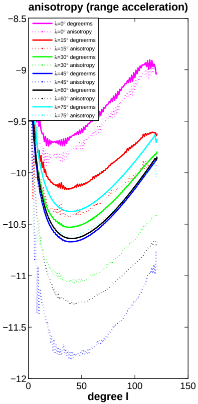


Fig: $\beta/\alpha = 490/31$, $\rho_{max} = 200\,\mathrm{km}$, $h = 308\,\mathrm{km}$ and $psd = 10^{-6}\,\mathrm{m/s^2/\sqrt{Hz}}$

- most homogeneous error pattern for $\lambda \approx 45^\circ$
- smallest formal errors $45^\circ - 60^\circ$

Summary

- (optimistic) error assessment by semi-analytical approach
- orbit perturbations via Hill differential equation
- special problem: time-dependent opening angle
  - Fourier series (with closed formulas)
  - convolution
- optimal angle depends on observation type
- spectral pattern of minimal formal error for
  - low degree
  - near sectorials