Do we need a new definition of gravity anomalies?

Torsten Mayer-Gürr and Christian Pock

WG Theoretical Geodesy and Satellite Geodesy
Institute of Geodesy, NAWI Graz
Graz University of Technology
Goal: Geoid determination in Austria with cm accuracy
Input data: Gravimetric observations

72,723 observations in and around Austria

Accuracy for most data: $<< 0.1 \text{ mGal}$
The fundamental equation of physical geodesy

\[
\Delta g = - \frac{\partial T}{\partial r} - 2 \frac{T}{r}
\]

Is the **fundamental equation of physical geodesy** in linearized form (with spherical approximation) **accurate enough** to model the observations?

Helmut Moritz (1980), Advanced physical geodesy:

“This spherical approximation causes an error which is negligible in most practical applications.”

“This error is [on a global average] on the order of ±20 cm in the geoidal height. This is one order of magnitude smaller than the accuracy implied by the present gravimetric and satellite data.”
Observation equations

Linearization of non-linear observation equations

\[ f(x) = f(x_0) + \frac{\partial f}{\partial x}\bigg|_{x_0} (x - x_0) + \cdots \]

Unknown parameter \( x = W(L, B, h) \)
Gravity potential

Taylor point \( x_0 = U(L, B, h) \)
Normal potential with \( N_0 = 0 \)

Difference \( x - x_0 = T(L, B, h) \)
Disturbance potential

Observed \( f(x) = g(L, B, h) \)
Observed absolute gravity

Computed \( f(x_0) = \gamma(L, B, h) \)
Normal gravity

ellipsoidal height = orthometric height + geoid height
\[ h = H + N(W) \]
Observation equations

Linearization of non-linear observation equations

\[ f(x) = f(x_0) + \left. \frac{\partial f}{\partial x} \right|_{x_0} (x - x_0) + \cdots \]

\[ \Delta g = -\frac{\partial T}{\partial r} - 2 \frac{T}{r} \]

Unknown parameter \( x = W(L, B, h) \)  
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Observed \( f(x) = g(L, B, h) \)  
Observed absolute gravity

Computed \( f(x_0) = \gamma(L, B, h_0) \)  
Normal gravity

Reduced observations
\[ f(x) - f(x_0) = g(L, B, H + N) - \gamma(L, B, H + N_0) = \Delta g \]

(Free air) gravity anomalies

Linear model \( \left. \frac{\partial f}{\partial x} \right|_{x_0} (x - x_0) = -\frac{\partial T}{\partial r} - 2 \frac{T}{r} \)  
Fundamental equation in physical geodesy
(In spherical approximation)
Observation equations

Linearization of non-linear observation equations

\[ f(x) = f(x_0) + \frac{\partial f}{\partial x}(x - x_0) + \cdots \]

Possible improvements of accuracy:

1. Better linearization (without spherical approximation)
Observation equations

Linearization of non-linear observation equations

\[ f(x) = f(x_0) + \frac{\partial f}{\partial x} \bigg|_{x_0} (x - x_0) + \cdots \]

Possible improvements of accuracy:

1. Better linearization (without spherical approximation)
2. Include quadratic terms
Observation equations

Linearization of non-linear observation equations

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Possible improvements of accuracy:

1. Better linearization (without spherical approximation)
2. Include quadratic terms
3. Better Taylor point
Observation equations

Linearization of non-linear observation equations

\[ f(x) = f(x_0) + \left. \frac{\partial f}{\partial x} \right|_{x_0} (x - x_0) + \cdots \]

Possible improvements of accuracy:

1. Better linearization (without spherical approximation)
2. Include quadratic terms
3. Better Taylor point
Taylor point

Approximate values (Taylor point)

Classic

\[ f(x_0) = \gamma(L, B, H + N_0) \quad \text{with} \quad N_0 = 0 \]

New approach

\[ f(x_0) = g_{sat}(L, B, H + N_{sat}) \]

Geoid

Satellite model GOCO05s
Taylor point

Approximate values (Taylor point)

Classic
\[ f(x_0) = \gamma(L, B, H + N_0) \quad \text{with} \quad N_0 = 0 \]

New approach
\[ f(x_0) = g_{sat}(L, B, H + N_{sat}) = \|g_{sat}\| = \|\nabla W_{sat}\| \]

Gravity potential:
\[ W(\lambda, \vartheta, r) = V(\lambda, \vartheta, r) + Z(\lambda, \vartheta, r) \]

Gravitational potential:
\[ V(\lambda, \vartheta, r) = \frac{GM}{R} \sum_{n=0}^{n_{\text{max}}} \left( \frac{R}{r} \right)^{n+1} \sum_{m=0}^{2n+1} c_{nm} C_{nm}(\lambda, \vartheta) + s_{nm} S_{nm}(\lambda, \vartheta) \]

Centrifugal potential:
\[ Z(\vartheta, r) = \frac{1}{2} \omega^2 r \sin \vartheta \]

Gravity:
\[ g = \|\nabla W\| = \left( \begin{array}{c} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \\ \frac{\partial W}{\partial z} \end{array} \right) = \left( \begin{array}{c} \frac{\partial W}{\partial x} r \partial \vartheta \\ \frac{\partial W}{\partial y} r \sin \vartheta \partial \lambda \\ \frac{\partial W}{\partial z} \partial \vartheta \end{array} \right) \]
Reduced observations

Classic approach

\[ \Delta g = g - \gamma \]

New approach

\[ \Delta g = g - g_{sat} \]
Reduced observations

Classic approach

\[ \Delta g = g - \gamma - \Delta g_{\text{sat}} \]

New approach

\[ \Delta g = g - \| g_{\text{sat}} \| \]
Reduced observations

Classic approach

\[ \Delta g = g - \gamma - \Delta g_{sat} \]

New approach

\[ \Delta g = g - \|g_{sat}\| \]
Reduced observations

Classic approach

$$\Delta g = g - \gamma - \Delta g_{\text{sat}}$$

New approach

$$\Delta g = g - \left\| g_{\text{sat}} \right\|$$
Gravitational potential from topographic masses

\[ T(\mathbf{r}_p) = G \iiint_{\Omega} \frac{1}{l(\mathbf{r}_p, \mathbf{r}_Q)} \rho(\mathbf{r}_Q) \, d\Omega(\mathbf{r}_Q) \]

Satellite model includes topographic effect

⇒ Spherical harmonic expansion of topography

\[ T_{\text{topo}}(\mathbf{r}_p) = \frac{GM}{R} \sum_{n=0}^{\infty} \left( \frac{R}{r} \right)^{n+1} \sum_{m=-n}^{n} a_{nm}^{\text{topo}} Y_{nm}(\mathbf{r}_p) \]

\[ a_{nm}^{\text{topo}} = \frac{1}{M (2n + 1)} \iiint_{\Omega} \left( \frac{r'}{R} \right)^n \rho(\mathbf{r}_Q) \, Y_{nm}(\mathbf{r}_Q) \, d\Omega(\mathbf{r}_Q) \]

Topography without satellite model part

\[ \delta T_{\text{topo}} = T_{\text{topo}} - \sum_{n=0}^{N} T_{n}^{\text{topo}} \]
Reduced observations

Classic approach
\[ \Delta g = g - \gamma - \Delta g_{\text{sat}} \]

New approach
\[ \Delta g = g - \| g_{\text{sat}} \| \]
Reduced observations

Classic approach

\[ \Delta g = g - \gamma - \Delta g_{sat} - \delta g_{topo} \]

New approach

\[ \Delta g = g - \left\| g_{sat} + g_{topo} \right\| \]
Reduced observations

Classic approach

\[ \Delta g = g - \gamma - \Delta g_{\text{sat}} - \Delta g_{\text{topo}} \]

New approach

\[ \Delta g = g - \left| g_{\text{sat}} + g_{\text{topo}} \right| \]
Reduced observations

Classic approach

$\Delta g = g - \gamma - \Delta g_{sat} - \delta g_{topo}$

New approach

$\Delta g = g - \| g_{sat} + g_{topo} \|$
Reduced observations

Classic approach

\[ \Delta g = g - \gamma - \Delta g_{\text{sat}} - \delta g_{\text{topo}} \]

New approach

\[ \Delta g = g - \| g_{\text{sat}} + g_{\text{topo}} \| \]
Estimated Geoid

Classic approach

\[ \Delta g = g - \gamma - \Delta g_{sat} - \delta g_{topo} \]

\[ \Rightarrow \text{Residual geoid} \]

New approach

\[ \Delta g = g - \left\| g_{sat} + g_{topo} \right\| \]

\[ \Rightarrow \text{Residual geoid} \]
Estimated Geoid

Classic approach
\[ \Delta g = g - \gamma - \Delta g_{sat} - \Delta g_{topo} \]

New approach
\[ \Delta g = g - \| g_{sat} + g_{topo} \| \]

Difference
Summary

Classical definition:
Gravity anomalies = Observed absolute gravity - Normal gravity
\[ \Delta g = g - \gamma \]

\[ \Rightarrow \text{Not accurate enough for geoid computation} \]

Generalized definition:
Gravity anomalies = Observed absolute gravity - Computed gravity from an approximate model
\[ \Delta g = g - \| \mathbf{g}_{sat} \| \]