Spheroidal and Ellipsoidal Harmonic Expansions of the Gravitational Potential of Small Solar System Bodies

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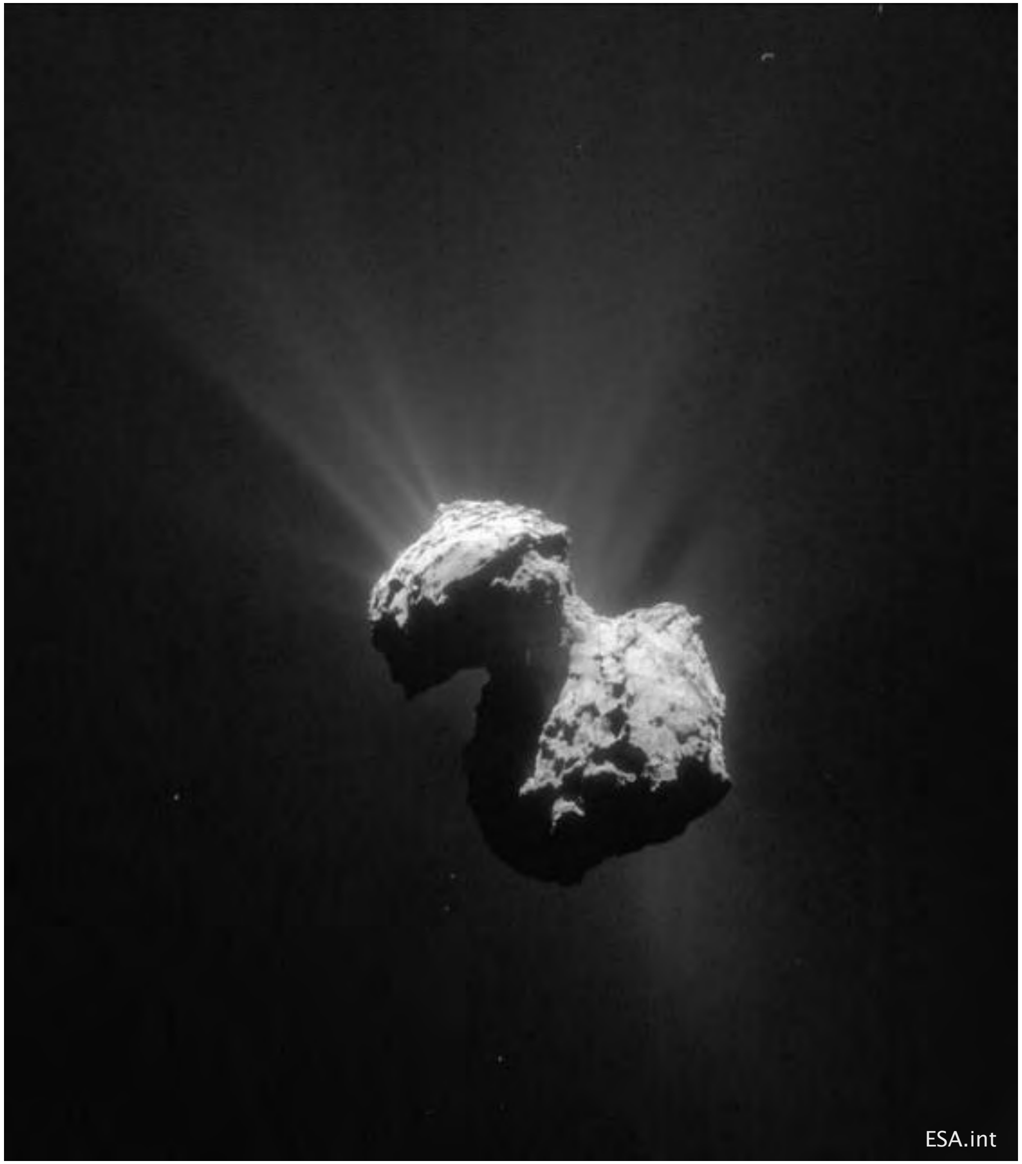
Rosetta: exploring comet 67P



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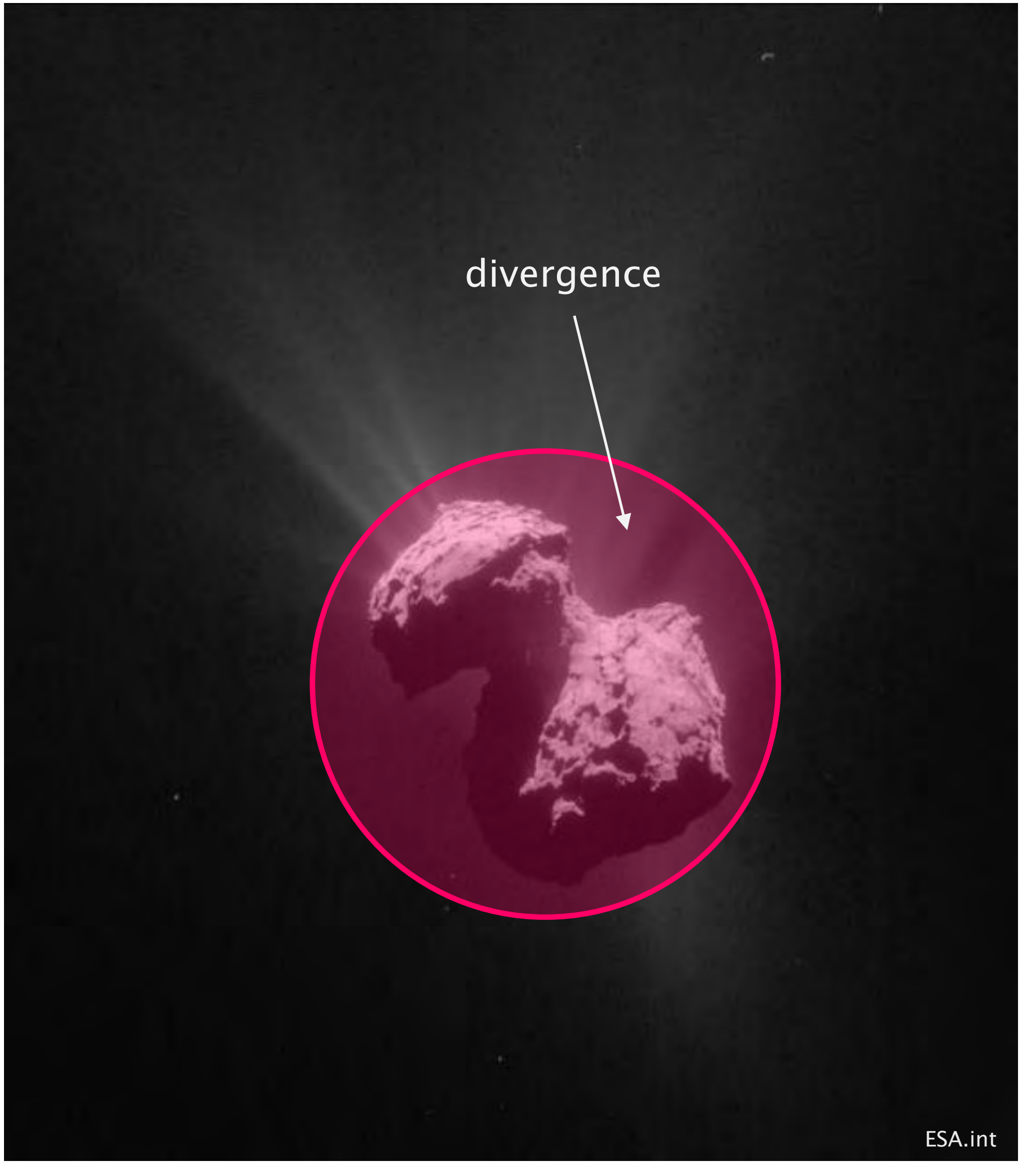
Gravitational field modeling

- Spherical harmonics
- Converge outside Brillouin sphere
- Bad geometrical fit



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Gravitational field modeling

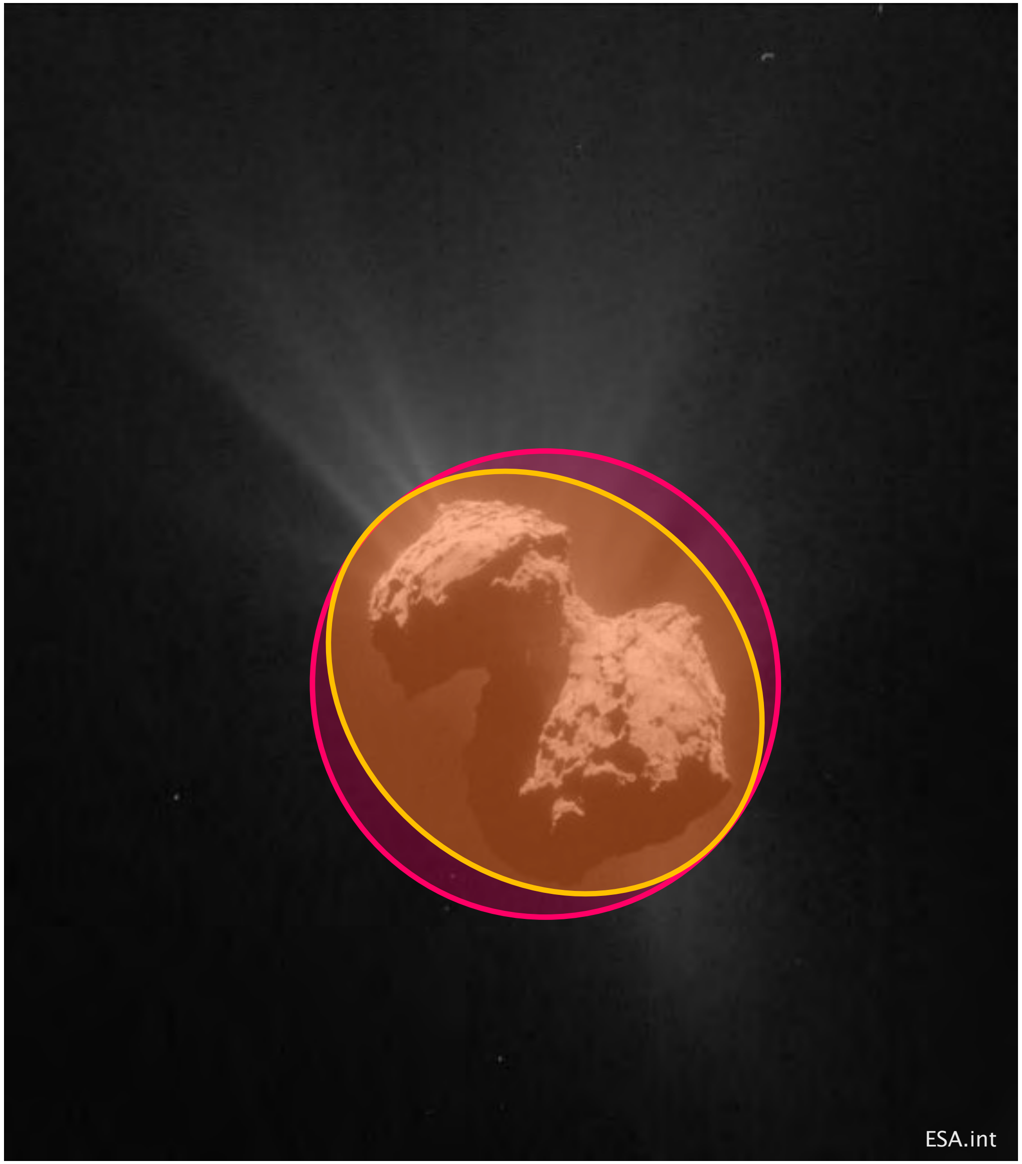
- Spherical harmonics
- Converge outside Brillouin sphere
- Bad geometrical fit

Alternatives

- Spheroidal harmonics
- Ellipsoidal harmonics

Benefit

- Reduction of divergence zone
- Close range evaluations
- Fast convergence

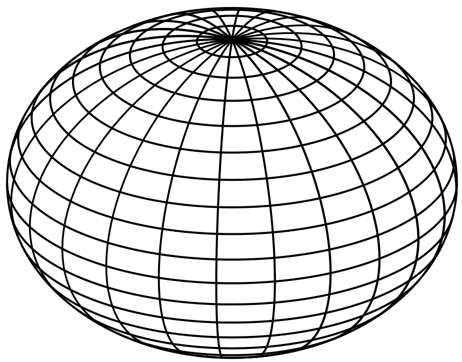


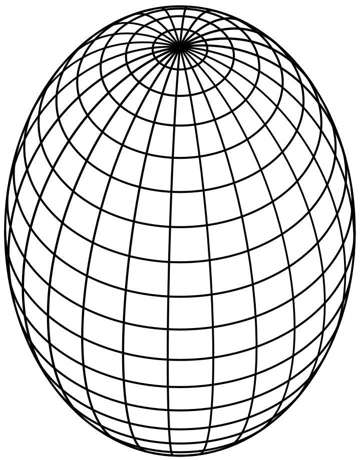
Spheroids and ellipsoids

Spheroids are ellipsoids of revolution

- Obtained by rotating an ellipse about one of its principal axes

oblate spheroid

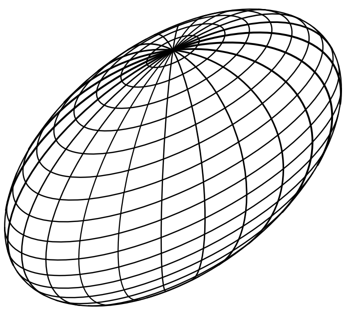




prolate spheroid

Ellipsoids are not surfaces of revolution

- The general ellipsoid has three distinct axes: $a > b > c$
- Sphere and spheroids are special cases of ellipsoids



Harmonic expansions of the potential

General form of Laplace‘s equation

- Arbitrary curvilinear and orthogonal coordinates ξ_1, ξ_2, ξ_3

$$\Delta V(\xi_1, \xi_2, \xi_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \xi_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial V}{\partial \xi_2}\right) + \frac{\partial}{\partial \xi_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial \xi_3}\right)\right] = 0$$

Harmonic expansions of the potential

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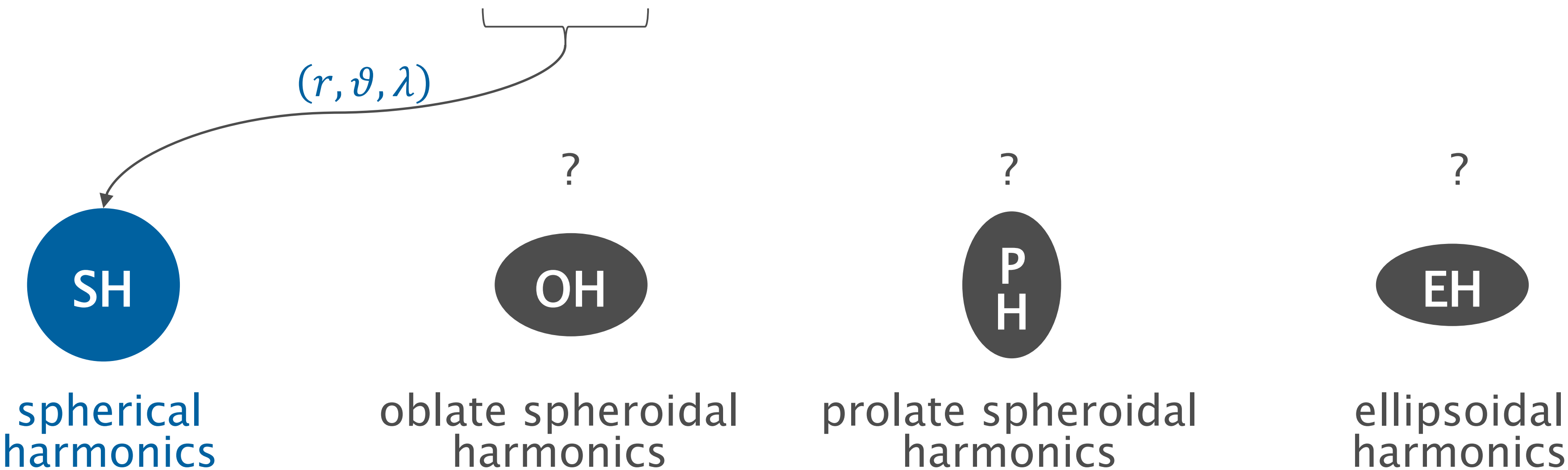
- Solution via separation of variables

$$V(\xi_1,\xi_2,\xi_3) = f(\xi_1)\times g(\xi_2)\times h(\xi_3)$$



Harmonic expansions of the potential

OH

Oblate spheroidal harmonics

- Semi–minor axis u, reduced colatitude ζ, longitude λ
- Reference spheroid a, b

$$V(u,\zeta,\lambda) = f(u) \times g(\zeta) \times h(\lambda) =$$

$$= \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{Q_{nm}(i\,u/\varepsilon)}{Q_{nm}(i\,b/\varepsilon)} \bar{P}_{nm}(\cos\zeta)\ \left[\bar{c}_{nm} \cos m\lambda + \bar{s}_{nm} \sin m\lambda\right]$$

associated Legendre functions of the second kind (ALF2)	ALF1	sinusoidal part

Harmonic expansions of the potential

P
H

Prolate spheroidal harmonics

- Semi-major axis v, reduced colatitude ζ, longitude λ
- Reference spheroid a, b

$$V(u,\zeta,\lambda) = f(v) \times g(\zeta) \times h(\lambda) =$$

$$= \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{Q_{nm}(i\, v/\varepsilon)}{Q_{nm}(i\, a/\varepsilon)} \bar{P}_{nm}(\cos\zeta) \; [\bar{c}_{nm} \cos m\lambda + \bar{s}_{nm} \sin m\lambda]$$

associated Legendre functions of the second kind (ALF2)

ALF1

sinusoidal part

Harmonic expansions of the potential

EH

Ellipsoidal harmonics

- Distance coordinate ρ, angular coordinates μ, ν
- Reference ellipsoid a, b, c

$$V(\rho,\mu,\nu) = f(\rho) \times g(\mu) \times h(\nu) =$$

$$= GM \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \bar{\alpha}_{nm} \frac{F_{nm}(\rho)}{F_{nm}(a)} \bar{E}_{nm}(\mu) \; \bar{E}_{nm}(\nu)$$

Lamé functions of the second kind

Lamé functions of the first kind

Harmonic expansions of the potential

EH

Ellipsoidal harmonics

- Computation of Lamé functions is not straightforward
  - Ambiguous coordinate transformation
  - No recursion formulas
  - Numerical integration of E_{nm} yields F_{nm}

- Numerical issues of higher degree functions
  - Close roots
  - Overflow issues

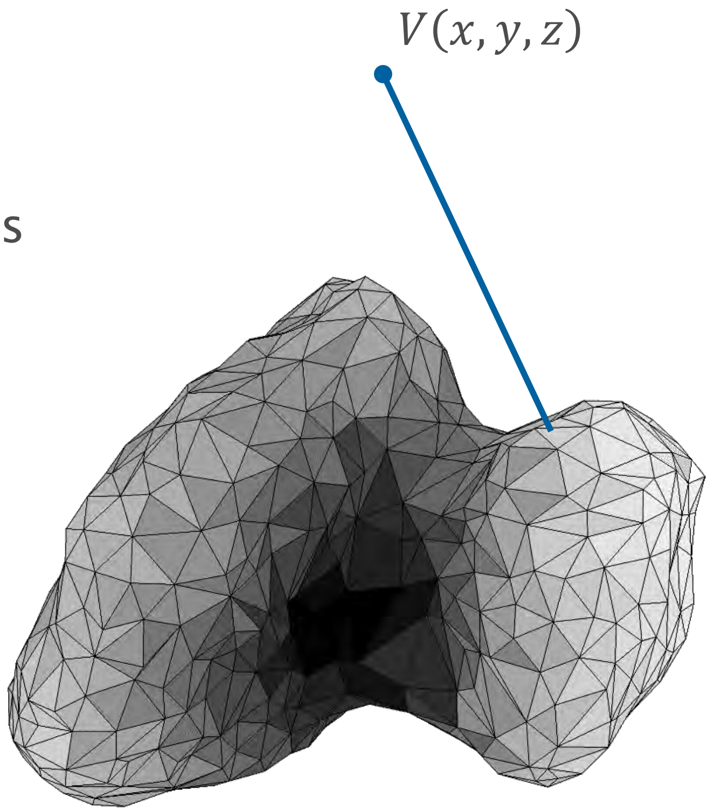
→ max. computable degree: ~20

Closed–loop simulation

Overview of simulation process

1. Simplification of the body
2. Analytical computation of gravitation
3. Estimation of harmonic series coefficients
4. Synthesis
5. Comparison with analytical values

$$\delta V = \frac{V_{harmonic} - V}{V} \times 100$$

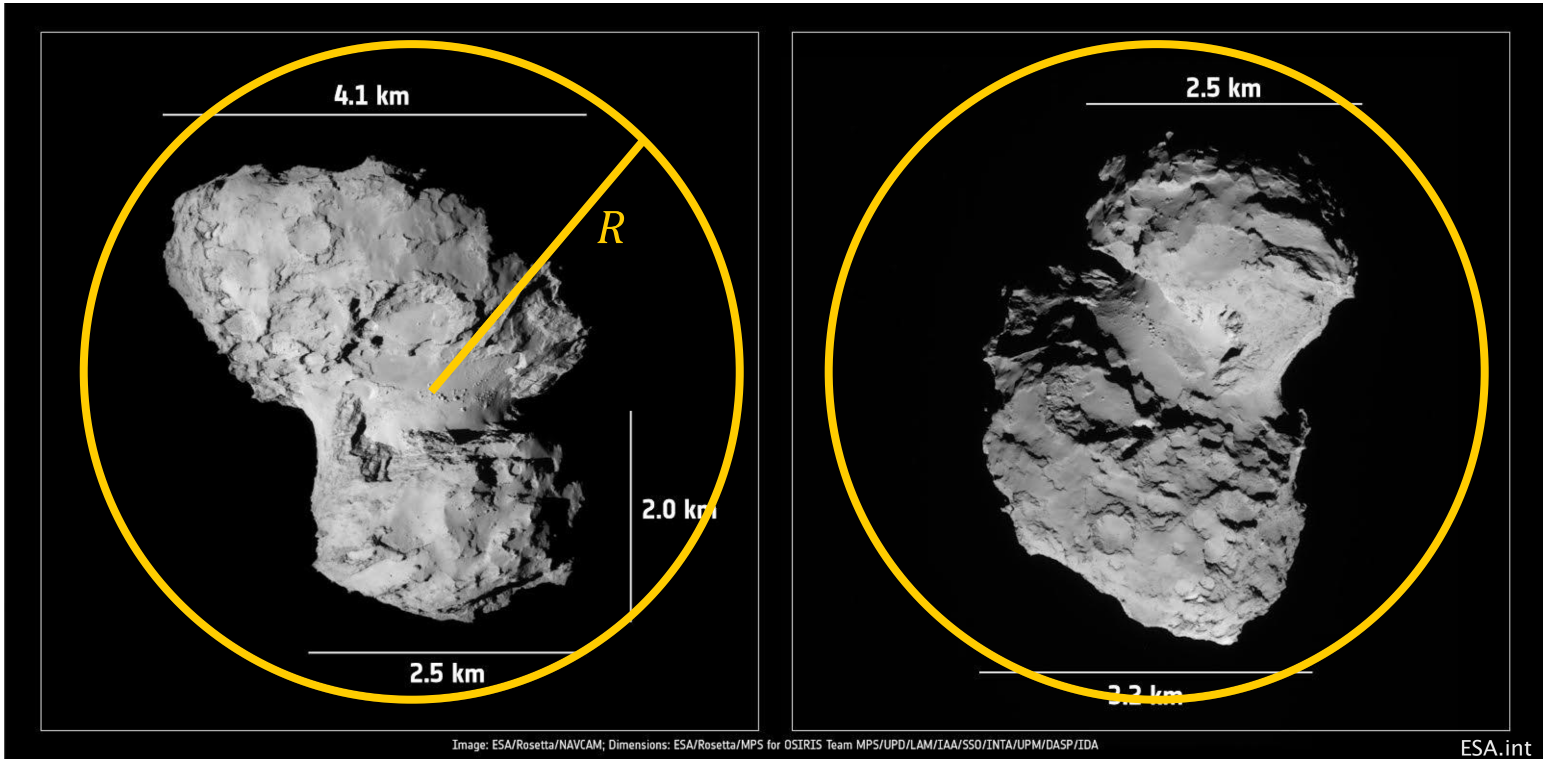


Polyhedral shape models
- NASA Planetary Data System (PDS)
- ESA website
- DAMIT database

Case studies

Comet 67P/Churyumov–Gerasimenko

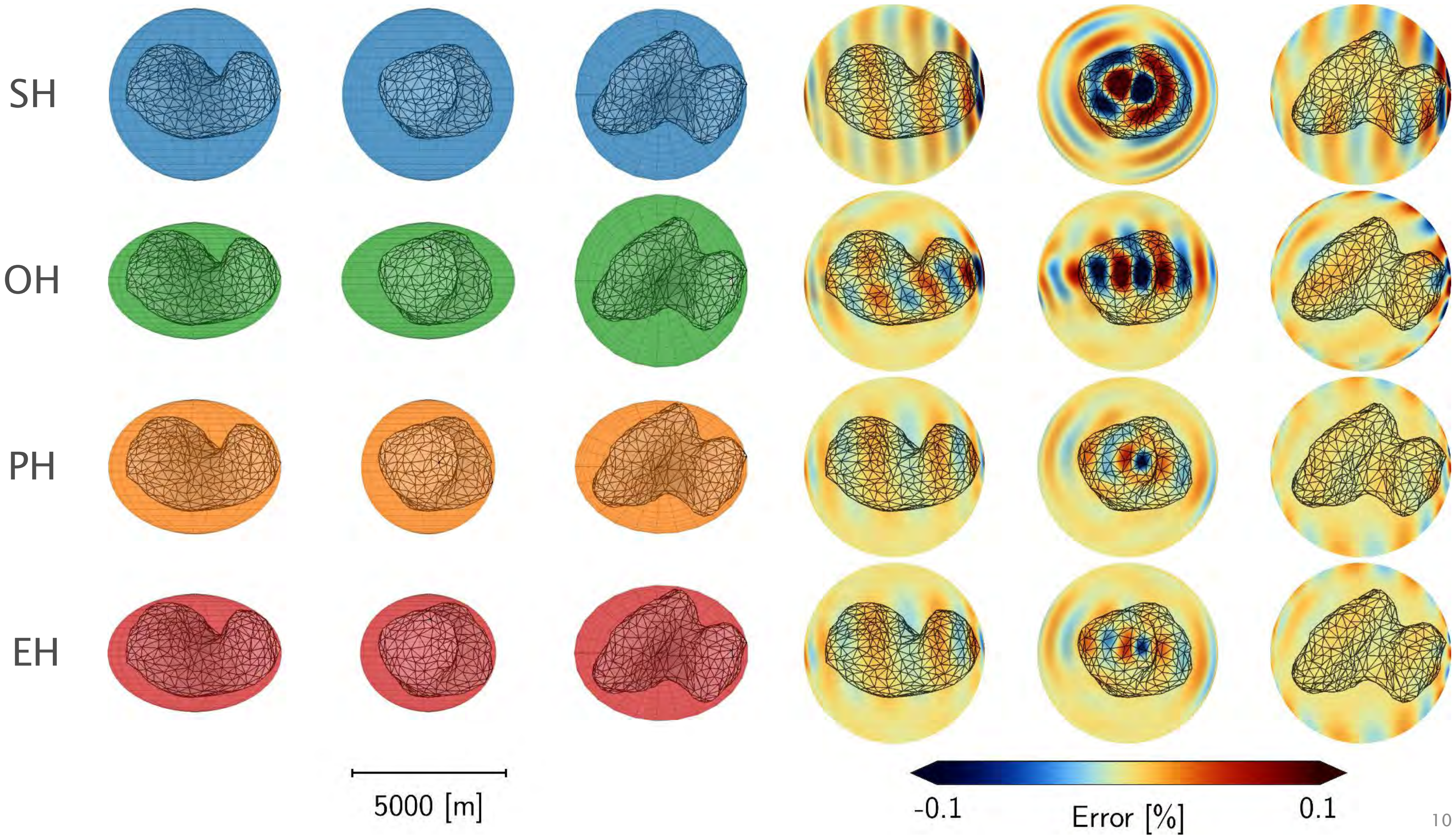
- Highly irregular in shape
- Very weak gravitational field (approx. 20 mgal at the surface)
- Evaluation sphere: $R = 3000\ \mathrm{m}$



Case studies

Comet 67P/Churyumov–Gerasimenko

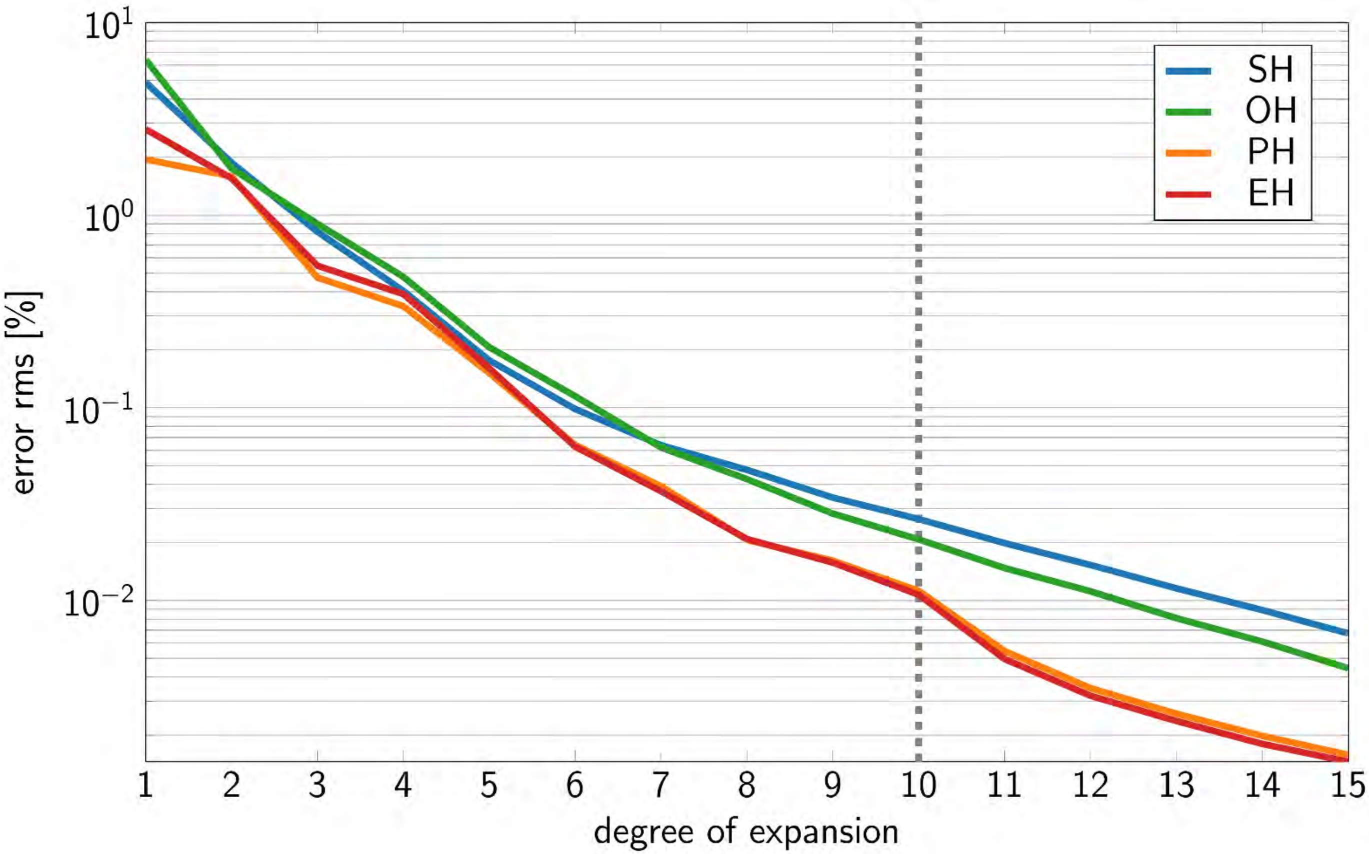
- Expansion degree $N = 10$



Case studies

Comet 67P/Churyumov–Gerasimenko

- Comparison of $\mathrm{rms}\,\delta V$ for varying degrees



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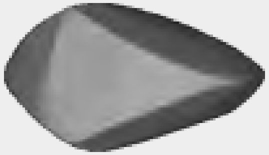
Case studies

Asteroids from the DAMIT database

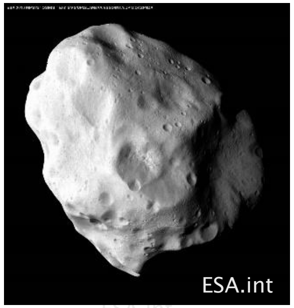
- Simulations for some 350 asteroids
- Comparison of spherical and spheroidal with ellipsoidal solutions
- Expansion degree $N = 10$
- Relative differences of rms values of percentage errors

$$\Delta\delta V = \frac{\mathrm{rms}\ \delta V - \mathrm{rms}\ \delta V_{EH}}{\mathrm{rms}\ V_{EH}}$$



D A M I T

Database of **A**steroid **M**odels from **I**nversion **T**echniques



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Case studies

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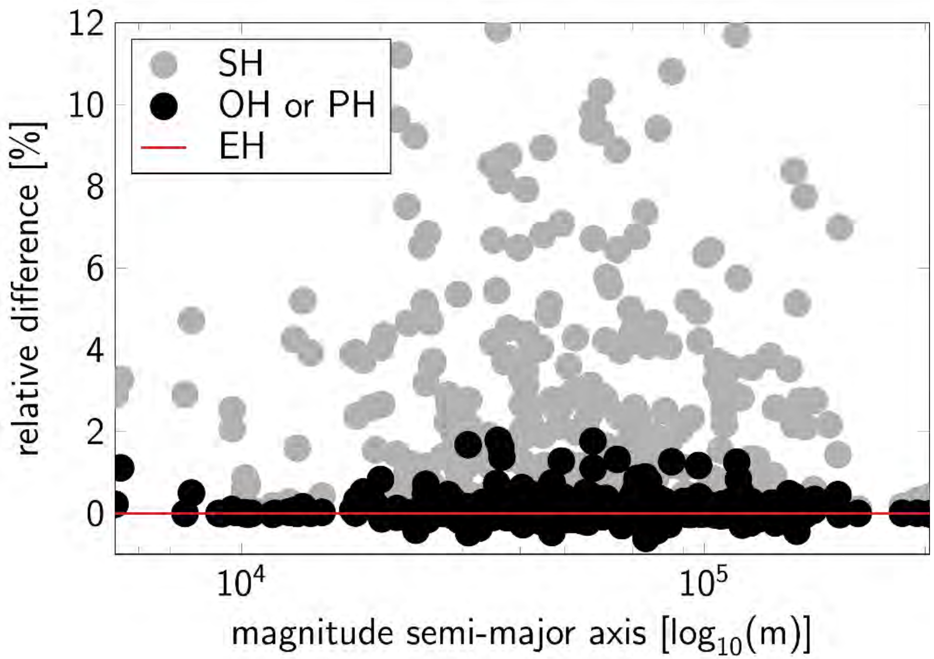
$$\Delta\delta V = \frac{\mathrm{rms}\ \delta V - \mathrm{rms}\ \delta V_{EH}}{\mathrm{rms}\ V_{EH}}$$

- 95% of spherical solutions within *ten* percent discrepancy
- 95% of spheroidal solutions within *one* percent discrepancy



Summary

Spheroidal harmonics

- Surface harmonics follow the same structure as spherical harmonics
- Exterior solutions are based on the second-kind associated Legendre functions

Ellipsoidal harmonics

- Based on the Lamé functions
- Decisively more demanding, both mathematically and numerically

Gravitational field modeling

- Reduction of divergence zone, close range evaluations
- Fast convergence of ellipsoidal harmonics
- Spheroidal harmonics almost equally well

References

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